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MATHEMATICAL METHODS OF QUANTUM OPTICS

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This book provides an accessible introduction to the mathematical methods of quantum optics. Starting from first principles, it reveals how a given system of atoms and a field is mathematically modelled. The method of eigenfunction expansion and the Lie algebraic method for solving equations are outlined. Analytically exactly solvable classes of equations are identified. The text also discusses consequences of Lie algebraic properties of Hamiltonians, such as the classification of their states as coherent, classical or non-classical based on the generalized uncertainty relation and the concept of quasiprobability distributions. A unified approach is developed for determining the dynamics of two-level and a three- level atom in combinations of quantized fields under certain conditions. Simple methods for solving a variety of linear and nonlinear dissipative master equations are given.

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Dedicated to
My Inspiration – My Wife
Shyama

Preface

This book is intended to provide a much needed systematic exposition of the mathematical methods of quantum optics, something that is not found in existing books. It is primarily addressed to researchers who are new to the field. The emphasis, therefore, is on a simple and self-contained, yet concise, presentation. It provides a unified view of the concepts and the methods of quantum optics and aims to prepare a reader to handle specific situations. A number of formulae scattered throughout the scientific literature are also brought together in a natural manner.

The broad plan of the book is to introduce first the basic physics and mathematical concepts, then to apply them to construct the model hamiltonians of the atom–field interaction and the master equation for an atom–field system interacting with the environment, and to analyze the equations so obtained. A brief description of the contents of the chapters is as follows.

The first chapter introduces the basic postulates of quantum mechanics, brings out their implications and develops the associated operational techniques. It discusses the measurement problem, the paradoxes of quantum mechanics and the local hidden variables theory, since quantum optics provides experimental means of examining these issues. Chapter 2 outlines the algebra of the exponential operator, which plays a prominent role in mathematical physics. The concept of Lie algebra is introduced and the standard hamiltonians of quantum optics are treated as elements of one or the other finite-dimensional Lie algebra. The question of representations of Lie algebras is addressed in Chap. 3. The notion of coherent states emerges as a continuous representation of a Lie algebra. The concept of quasiprobabilities is developed in Chap. 4. Their usefulness as operational tools and as entities for identifying purely quantum effects is demonstrated. Chapter 5 presents the essential elements of the theory of stochastic processes. The theory of classical and quantized electromagnetic (e.m.) fields is outlined in Chap. 6. It describes the characterization of the e.m. field in terms of its correlation functions and also their role in identifying the signatures of field quantization.

By starting with the hamiltonian for an atom interacting with the e.m. field in the dipole approximation, Chap. 7 describes ways of reducing it to simpler, mathematically tractable forms commensurate with given physical conditions. The standard models of quantum optics are thereby derived. The

effects of the environment on an atom–field system are the subject of the quantum theory of damping outlined in Chap. 8. Here the master equation for the evolution of a system in contact with a reservoir is constructed and methods of solving it are discussed.

Chapter 9 analyzes the perturbative solution of the master equation of an atomic system in an external field. This leads to the notions of susceptibility, multiwave mixing and the absorption spectrum.

The method of solving a set of linear equations with time-independent coefficients in terms of generalized eigenvectors is outlined in Chap. 10. That chapter presents the solution of a two-term recurrence relation and identifies and solves exactly solvable quadratic three-term recurrence relations. These recurrence relations encompass many well-known quantum optical situations.

Chapters 11–14 deal with the solution of some standard model systems. Chapter 11 identifies the class of analytically exactly solvable models of an effective two-level atom and that of an effective three-level atom in a quantized field. It provides a unified treatment of the exactly solvable hamiltonians of quantum optics.

The problem of an externally driven two-level atomic system dissipating into a squeezed reservoir is addressed in Chap. 12. The exactly solvable cases of an arbitrary time-dependent drive are identified. The exact dynamics in a monochromatic drive is investigated and the collective effects in a driven two-level atomic system are highlighted. Chapter 12 also briefly discusses the dynamical behaviour of a three-level atom dissipating into a reservoir at absolute zero temperature and reveals the effects of almost equally spaced pairs of energy levels.

The dynamics of a field dissipating into a linear or two-photon non-linear reservoir is the subject of Chap. 13. The evolution of an atomic system interacting with a single damped quantized cavity mode is investigated in Chap. 14. This chapter also outlines the theory of the micromaser.

I am indebted to Girish Agarwal for teaching me the subject of quantum optics. Valuable contributions to my understanding have been gained through my association with Robert (Robin) Bullough, Joseph Eberly, Fritz Haake, Shoukry Hassan, Rajiah Simon, Subhash Chaturvedi, V. Srinivasan, Subhasish Dattagupta, Surya Tiwari, Dinkar Khandekar and Suresh Lawande. I am grateful to Debabrata Biswas and Aditi Ray for their valuable suggestions and help in preparing the manuscript. I am thankful to Dinesh Sahni for his support and encouragement. Angela Lahee of Springer-Verlag deserves a big thank you for her careful editing.

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1. Basic Quantum Mechanics

Quantum optics is the quantum theory of interaction of the electromagnetic field with matter. In this chapter we recapitulate basic concepts and operational methods of the quantum theory essential for developing the theory of quantum optics. We delve also in to the controversial issue of interpretation of the quantum theory as a classical statistical theory. Quantum optics provides means for subjecting these conceptually controversial issues to experimental tests.

1.1 Postulates of Quantum Mechanics

In this section we state five basic postulates of Quantum Mechanics and discuss some of their important implications.

1.1.1 Postulate 1

An isolated quantum system is described by a vector in a Hilbert space. Two vectors differing only by a multiplying constant represent the same physical state.

Following the notation introduced by Dirac [1], we represent a vector by a *ket*, $|\ \rangle$.

A Hilbert space is a complex linear vector space equipped with the definition of a scalar product and spanned by a complete set of vectors [2]. The meaning and implications of these properties of the Hilbert space are explained below. They are crucial for relating the theory with experimental observations.

Linear Vector Space. A Hilbert space is a complex linear vector space. We assume familiarity with the notion of a linear vector space over the field of complex numbers (c-numbers) [2]. We recall that if $|\psi_1\rangle$ and $|\psi_2\rangle$ are vectors in a complex linear vector space then a *linear combination* $\alpha_1|\psi_1\rangle + \alpha_2|\psi_2\rangle$ for arbitrary complex numbers α_1, α_2 is also a vector in the same space. A set of vectors $|\psi_1\rangle, \cdots, |\psi_n\rangle$ is said to be *linearly independent* if

$$\sum_{i=i}^{n} \alpha_i |\psi_i\rangle = 0, \tag{1.1}$$

implies $\alpha_i = 0$ for all $i = 1, \ldots, n$. The maximum number of linearly independent vectors in a linear vector space is called its *dimension.*

Scalar Product. To say that the Hilbert space is a *Euclidean or scalar product space* means that it is possible to associate with every pair of vectors $|\phi\rangle$ and $|\psi\rangle$ in it a complex number, denoted by $\langle\phi|\psi\rangle$, such that

1. $\langle\phi|\psi\rangle = \langle\psi|\phi\rangle^*$, where $*$ denotes the operation of complex conjugation;
2. If $|\psi\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle$ then $\langle\phi|\psi\rangle = c_1\langle\phi|\psi_1\rangle + c_2\langle\phi|\psi_2\rangle$;
3. $\langle\psi|\psi\rangle > 0$;
4. $\langle\psi|\psi\rangle = 0$ if and only if (iff) $|\psi\rangle = 0$.

In the following we list some consequences of these axioms.

- The scalar product associates with a vector $|\ \rangle$ its *dual* $\langle\ |$ called a *bra* [1].
- The non-zero positive number $\big|\big||\psi\rangle\big|\big| \equiv \sqrt{\langle\psi|\psi\rangle}$ is called the *norm* or the *length* of the vector. Since two vectors differing only by a multiplication factor represent the same physical state, we can represent a physical state by a vector of a fixed, say unit, norm if the norm is finite. Hence, $|\psi\rangle$ is physically an acceptable vector if its norm is finite i.e. if

$$\langle\psi|\psi\rangle < \infty. \tag{1.2}$$

- The vector $|\phi\rangle\langle\phi|\psi\rangle$ is the *projection* of a vector $|\psi\rangle$ along the vector $|\phi\rangle$. The scalar product $\langle\phi|\psi\rangle$ is a measure of the overlap between the vectors $|\psi\rangle$ and $|\phi\rangle$. If $\langle\phi|\psi\rangle = 0$ then $|\psi\rangle$ and $|\phi\rangle$ are said to be *orthogonal* to each other.
- Two sets of vectors $|\psi_1\rangle, \cdots, |\psi_n\rangle$ and $|\phi_1\rangle, \cdots, |\phi_n\rangle$ are said to be orthonormal to each other if

$$\langle\phi_i|\psi_j\rangle = \delta_{ij}, \qquad i, j = 1, \ldots, n. \tag{1.3}$$

- A set $|e_1\rangle, \cdots |e_n\rangle$ of vectors is said to be orthonormal if

$$\langle e_i|e_j\rangle = \delta_{ij}, \qquad i, j = 1, \ldots, n. \tag{1.4}$$

- An important consequence of the axioms defining the scalar product is the *Schwarz inequality*

$$\langle\phi|\phi\rangle\langle\psi|\psi\rangle \geq \langle\phi|\psi\rangle\langle\psi|\phi\rangle, \tag{1.5}$$

where the equality holds if and only if the two vectors in question are *linearly dependent* i.e. if

$$|\psi\rangle = \mu|\phi\rangle, \tag{1.6}$$

μ being a complex number. In order to establish this, show that the minimum value of $\langle\Psi(\mu)|\Psi(\mu)\rangle$, where $|\Psi\rangle = |\psi\rangle - \mu|\phi\rangle$, as a function of μ is $\langle\psi|\psi\rangle - |\langle\psi|\phi\rangle|^2/\langle\phi|\phi\rangle$. The requirement that this value, due to axiom 3 of the scalar product, be positive leads to the Schwarz inequality in (1.5). Also, according to the axiom 4 above, $\langle\Psi(\mu)|\Psi(\mu)\rangle = 0$ iff $|\Psi(\mu)\rangle = 0$ i.e.

iff (1.6) holds. It may be verified easily that (1.5) then holds with equality. In a similar way we can derive the generalized Schwarz inequality

$$\det(\langle\psi_\mu|\psi_\nu\rangle) \geq 0, \tag{1.7}$$

where $\det(\langle\psi_\mu|\psi_\nu\rangle)$ is the determinant of the matrix constituted by the elements $\langle\psi_\mu|\psi_\nu\rangle$, $\mu, \nu = 1, \ldots, n$. Invoking the fact that the determinant of a matrix is zero if its rows (or columns) are linearly dependent, it follows that the equality in (1.7) holds iff $|\psi_\mu\rangle$ are linearly dependent.

Completeness. In a scalar product vector space of finite dimension n, there always exists a set of n linearly independent vectors $\{|\psi_i\rangle\}$, called the *basis* vectors, such that any vector $|\psi\rangle$ can be expressed as a linear combination [2],

$$|\psi\rangle = \sum_{i=1}^{n} d_i|\psi_i\rangle. \tag{1.8}$$

The complex numbers $\{d_i\}$ in a scalar product space may be determined by taking the scalar product of (1.8) with the vectors $\{|\phi_i\rangle\}$ orthonormal to $\{|\psi_i\rangle\}$ to give $d_i = \langle\phi_i|\psi\rangle$ so that

$$|\psi\rangle = \sum_{i=1}^{n} |\psi_i\rangle\langle\phi_i|\psi\rangle. \tag{1.9}$$

The vector $|\psi\rangle$ in an n-dimensional space is thus characterized by n complex numbers $\{\langle\phi_i|\psi\rangle\}$. The column of these numbers constitutes a *representation* of the vector in the given basis. The dual $\langle\psi|$ of $|\psi\rangle$ is then represented by the row constituted by the numbers $\{\langle\psi|\phi_i\rangle\} = \{\langle\phi_i|\psi\rangle^*\}$. Thus the representation of $\langle\psi|$ is obtained by the process of *hermitian conjugation* (interchanging of rows and columns along with the operation of complex conjugation), denoted by $\dagger$, of $|\psi\rangle$: $\langle\psi| = (|\psi\rangle)^\dagger$.

The expansion (1.9) in a scalar product space is guaranteed if the space is finite-dimensional. However, such an expansion need not exist if the space is infinite-dimensional. In quantum mechanics, we are concerned only with those scalar product linear vector spaces in which every vector is expressible in terms of a basis. Such a space is called a Hilbert space.

Now, on invoking the fact that (1.9) is to be valid for an arbitrary $|\psi\rangle$, follows the *completeness relation*

$$\sum_{i=1}^{n} |\psi_i\rangle\langle\phi_i| = I, \tag{1.10}$$

where I is the identity operator defined below. If $\{|\psi_i\rangle\}$ is orthonormal, i.e. if $\{|\phi_i\rangle\} = \{|\psi_i\rangle\} \equiv \{|e_i\rangle\}$, then (1.10) reduces to

$$\sum_{i=1}^{n} |e_i\rangle\langle e_i| = I. \tag{1.11}$$

In our discussion so far we have assumed that the basis vectors are denumerable. There are, however, occasions which require us to work with a basis labeled by a continuous parameter. Consider an orthonormal set of basis vectors $|\xi\rangle$ labeled by a real continuous parameter ξ. The condition of orthonormality then reads

$$\langle \xi' | \xi \rangle = \delta(\xi - \xi'), \tag{1.12}$$

$\delta(x)$ being the Dirac delta function. If $a \leq \xi \leq b$ then the analog of the expansion of a vector in terms of the basis vectors is

$$|\psi\rangle = \int_a^b \langle \xi | \psi \rangle \; |\xi\rangle \; \mathrm{d}\xi. \tag{1.13}$$

A vector $|\psi\rangle$ in a continuous basis is thus represented by the function $c(\xi) = \langle \xi | \psi \rangle$ of a real variable ξ.

Operators. The action of a force transforms the state of a system. A transformation of a state of a system may be described by a rule, called an *operator*, that associates with a vector in the space another vector in the same space. If, for example, an action transforms $|\psi\rangle$ to $|\phi\rangle$ then we write

$$\hat{A}|\psi\rangle = |\phi\rangle \tag{1.14}$$

where the operator $\hat{A}$ defines the rule of transformation. We distinguish an operator from a c-number variable by a caret on the former. An operator $\hat{A}$ is *linear* if, for any complex numbers c_1 and c_2,

$$\hat{A}\Big(c_1|\psi\rangle + c_2|\phi\rangle\Big) = \Big(c_1\hat{A}|\psi\rangle + c_2\hat{A}|\phi\rangle\Big). \tag{1.15}$$

We shall be concerned only with linear operators.

If $\hat{A}|\psi\rangle = |\psi\rangle$ for all $|\psi\rangle$ then $\hat{A}$ is called the *unit* or *identity* operator, often denoted by I. Since I acts like the scalar unity, we do not dress it with a caret and even denote it by 1.

In order to obtain a c-number representation of an operator $\hat{A}$, consider an orthonormal basis $\{|e_i\rangle\}$. Rewrite $\hat{A}$ as $I\hat{A}I$ where I is the unit operator and express I in terms the completeness relation (1.11) to get

$$\hat{A} = \sum_{i,j=1}^{n} \langle e_i | \hat{A} | e_j \rangle \; |e_i\rangle\langle e_j|. \tag{1.16}$$

The operator $\hat{A}$ may be represented by an $n \times n$ matrix constituted by the complex numbers $\langle e_i|\hat{A}|e_j\rangle$ $(i, j = 1, \ldots, n)$. On operating (1.16) on an arbitrary vector $|\psi\rangle$, it follows that $\hat{A}|\psi\rangle$ is represented by the product of the matrix $\langle e_i|\hat{A}|e_j\rangle$ representing $\hat{A}$ with the column $\langle e_j|\psi\rangle$ representing $|\psi\rangle$. It is straightforward also to show that a product $\hat{A}\hat{B}$ is represented by the product of the matrices representing them. Thus, the correspondence between vectors as columns and operators as matrices is not only notational but also operational.

The analog of (1.16) in the continuous representation is evidently

$$\hat{A} = \int_a^b \int_a^b \langle \xi | \hat{A} | \xi' \rangle \, |\xi\rangle\langle\xi'| \, d\xi \, d\xi'. \tag{1.17}$$

The function $a(\xi, \xi') = \langle \xi | \hat{A} | \xi' \rangle$ of two real variables ξ and ξ' now serves as a representation of $\hat{A}$. The rules of addition and multiplication of operators and those for the action of an operator on a vector are same as the corresponding ones for the discrete case with summations replaced by integrals.

Next, we enumerate some definitions and algebraic operations involving linear operators. The treatment, though may lack rigor at times, is adequate for our purpose. For details, see, e.g., [3].

1. The product $\hat{A}\hat{B}$ denotes the action of $\hat{B}$ on a state followed by that of $\hat{A}$ on the resulting state. This result need not be the same as that due to the operation $\hat{B}\hat{A}$. The operator defined by

   $$[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A} \tag{1.18}$$

   is called the *commutator* of $\hat{A}$ and $\hat{B}$. If $[\hat{A}, \hat{B}] = 0$ then $\hat{A}$ and $\hat{B}$ are said to *commute*.
2. If m is a positive integer then $\hat{A}^m$ denotes $\hat{A}$ multiplied with itself m times.
3. A function $F(\hat{A})$ of an operator $\hat{A}$ may be defined by its expansion in terms of the powers of $\hat{A}$. Of particular interest is the *exponential operator* defined by the expansion

   $$\exp(\hat{A}) = \sum_{m=0}^{\infty} \frac{1}{m!} \hat{A}^m. \tag{1.19}$$

   Using this definition, the reader should verify that if $[\hat{A}, \hat{B}] = 0$ then

   $$\exp[\hat{A} + \hat{B}] = \exp(\hat{A}) \exp(\hat{B}) = \exp(\hat{B}) \exp(\hat{A}). \tag{1.20}$$

   The problem of disentangling the exponential of a sum of non-commuting operators is addressed in Chap. 2.
4. Check that we can write $\hat{B}(\hat{A}\hat{B})^m = (\hat{B}\hat{A})\hat{B}(\hat{A}\hat{B})^{m-1} = \cdots = (\hat{B}\hat{A})^m \hat{B}$. As a consequence of this it follows that if $F(\hat{A}\hat{B})$ is a function expandable as a power series of its argument then

   $$\hat{B}F(\hat{A}\hat{B}) = F(\hat{B}\hat{A})\hat{B}. \tag{1.21}$$

5. If there exists a nonnegative real number C such that $\sqrt{|\langle \psi | \hat{A} | \psi \rangle|} \leq C\sqrt{\langle \psi | \psi \rangle}$ for all $|\psi\rangle$ then $\hat{A}$ is called a *bounded* operator. The minimum of the numbers C, denoted by $||\hat{A}||$, is called the *norm* of $\hat{A}$.
6. The *adjoint* of $\hat{A}$, denoted by $\hat{A}^\dagger$, is defined by the relation

   $$\langle \phi | \hat{A} | \psi \rangle = \langle \psi | \hat{A}^\dagger | \phi \rangle^*. \tag{1.22}$$

for all $|\psi\rangle$ and $|\phi\rangle$ in the given Hilbert space. Combine this with (1.16) to show that the matrix representing the adjoint of an operator is the adjoint of the matrix representing that operator. Verify also that

$$(\hat{A}\hat{B}\hat{C})^\dagger = \hat{C}^\dagger \hat{B}^\dagger \hat{A}^\dagger. \tag{1.23}$$

7. If, corresponding to an operator $\hat{A}$, there exists an operator $\hat{B}$ such that $\hat{B}\hat{A} = \hat{A}\hat{B} = I$ then $\hat{B}$ is called the *inverse* of $\hat{A}$. The inverse of $\hat{A}$ is denoted by $\hat{A}^{-1}$:

$$\hat{A}^{-1}\hat{A} = \hat{A}\hat{A}^{-1} = I. \tag{1.24}$$

An operator is called *singular* if it does not admit an inverse. It can be shown that $\hat{A}\hat{B} = I$ implies $\hat{B}\hat{A} = I$ in a finite dimensional Hilbert space but not if the space is infinite dimensional [3]. In an infinite dimensional space, an operator $\hat{A}$ may be singular but corresponding to it there may still exist an operator $\hat{A}_L^{-1}$ called the *left inverse* of $\hat{A}$ such that $\hat{A}_L^{-1}\hat{A} = I$ or an operator $\hat{A}_R^{-1}$ called its *right inverse* such that $\hat{A}\hat{A}_R^{-1} = I$. Clearly, an operator which is the right as well the left inverse of $\hat{A}$ is the inverse $\hat{A}^{-1}$. It is straightforward to show that

$$(\hat{A}\hat{B}\hat{C})^{-1} = \hat{C}^{-1}\hat{B}^{-1}\hat{A}^{-1}, \tag{1.25}$$

provided that $\hat{A}$, $\hat{B}$, and $\hat{C}$ are non-singular. Since an operator commutes with itself, it follows from (1.20) that the inverse of $\exp(\hat{A})$ is $\exp(-\hat{A})$. Notice also that

$$\left[\exp(\hat{A}_1)\cdots\exp(\hat{A}_m)\right]^{-1} = \exp(-\hat{A}_m)\cdots\exp(-\hat{A}_1). \tag{1.26}$$

8. An operator $\hat{H}$ is called *hermitian* if $\hat{H} = \hat{H}^\dagger$.
9. An operator $\hat{U}$ is called *unitary* if $\hat{U}\hat{U}^\dagger = \hat{U}^\dagger\hat{U} = I$. This shows that $\langle\psi_i|\psi_j\rangle = \langle\psi_i|\hat{U}^\dagger\hat{U}|\psi_j\rangle$. Hence the set of states $\{\hat{U}|\psi_i\rangle\}$ obtained by a unitary transformation of the set $\{|\psi_i\rangle\}$ preserves the scalar product. Hence, the set of vectors obtained as a result of unitary transformation of an orthonormal set of vectors is also orthonormal. If the orthonormal set is complete then so is the set obtained by its unitary transformation. Verify that $\hat{U}$ can be represented as $\hat{U} = \exp(\mathrm{i}\hat{H})$ where $\hat{H}$ is hermitian.
10. If an operator $\hat{A}$ commutes with its adjoint i.e. if $[\hat{A}, \hat{A}^\dagger] = 0$ then it is called a *normal* operator. Note that hermitian and unitary operators are examples of a normal operator.
11. If $\langle\psi|\hat{A}|\psi\rangle \geq 0$ for all $|\psi\rangle$ then $\hat{A}$ is said to be *positive*. Let $|\phi\rangle = \hat{B}|\psi\rangle$. Then $\langle\psi|\hat{B}^\dagger\hat{B}|\psi\rangle = \langle\phi|\phi\rangle \geq 0$. Hence $\hat{B}^\dagger\hat{B} \geq 0$ for all $\hat{B}$.
12. The sum of the diagonal elements of a matrix representing an operator $\hat{A}$ is called the *trace* of $\hat{A}$. It is often denoted by $\mathrm{Tr}(\hat{A})$. If $\{|e_i\rangle\}$ is an orthonormal basis then, by definition,

$$\mathrm{Tr}(\hat{A}) = \sum_{i=1}^{n}\langle e_i|\hat{A}|e_i\rangle. \tag{1.27}$$

The value of the trace of an operator is independent of the basis. Some consequences of the definition of trace are:

- The complex conjugate of (1.27), read with (1.22), shows that
$$\mathrm{Tr}(\hat{A}^\dagger) = \left[\mathrm{Tr}(\hat{A})\right]^* \tag{1.28}$$
- Invoke the definition of trace and completeness of $\{|e_i\rangle\}$ to show that
$$\mathrm{Tr}\left[|\psi\rangle\langle\phi|\right] = \sum_{i=1}^{n}\langle e_i|\psi\rangle\langle\phi|e_i\rangle = \langle\phi|\psi\rangle, \tag{1.29}$$
and
$$\mathrm{Tr}\left[|\psi\rangle\langle\phi|\hat{A}\right] = \sum_{i=1}^{n}\langle e_i|\psi\rangle\langle\phi|\hat{A}|e_i\rangle = \langle\phi|\hat{A}|\psi\rangle. \tag{1.30}$$
- Verify that the trace of a product of operators possesses the cyclic property:
$$\mathrm{Tr}(\hat{A}\hat{B}\hat{C}) = \mathrm{Tr}(\hat{C}\hat{B}\hat{A}). \tag{1.31}$$
- If $\hat{U}$ is unitary then, due to (1.31), $\mathrm{Tr}[\hat{U}^\dagger\hat{A}\hat{U}] = \mathrm{Tr}[\hat{U}\hat{U}^\dagger\hat{A}] = \mathrm{Tr}[\hat{A}]$. This shows that the trace of $\hat{A}$ and that of $\hat{U}^\dagger\hat{A}\hat{U}$ are equal.

13. If $|\psi\rangle$ is such that
$$\hat{A}|\psi\rangle = \lambda|\psi\rangle, \tag{1.32}$$
where λ is a constant then $|\psi\rangle$ is called an *eigenvector* or an *eigenstate* and λ the corresponding *eigenvalue* of $\hat{A}$. Expand $F(\hat{A})$ in powers of $\hat{A}$ and use (1.32) repeatedly to show that
$$F(\hat{A})|\psi\rangle = F(\lambda)|\psi\rangle. \tag{1.33}$$

The problem of solving an eigenvalue equation is addressed in Chap. 10. We recall from that chapter that

- The eigenvalues of a hermitian operator are real.
- Any normal operator in an n-dimensional space possesses n eigenvectors which are orthonormal. Hence, if $\hat{A}$ is a normal operator and $\{|a_i\rangle\}$ the set of its orthonormal eigenvectors then $\langle a_j|\hat{A}|a_i\rangle = a_i\delta_{ij}$. On combining this with (1.16) follows the expansion
$$\hat{A} = \sum_{i=1}^{n} a_i|a_i\rangle\langle a_i|. \tag{1.34}$$
of a normal operator in its eigenbasis. Apply (1.33) to show that
$$F\left(\hat{A}\right) = \sum_{i=1}^{n} F(a_i)|a_i\rangle\langle a_i|. \tag{1.35}$$

Now, let $|a,b\rangle$ be a simultaneous eigenvector of $\hat{A}$ and $\hat{B}$ such that
$$\hat{A}|a,b\rangle = a|a,b\rangle, \qquad \hat{B}|a,b\rangle = b|a,b\rangle. \tag{1.36}$$
Operate first (second) of the equations above by $\hat{B}$ ($\hat{A}$) and subtract the resulting equations to obtain
$$[\hat{A},\ \hat{B}]|a,b\rangle = 0. \tag{1.37}$$

This equation is trivially solved if $[\hat{A},\ \hat{B}] = 0$ showing that non-commuting operators may possess common eigenvectors. If $\hat{A}$ and $\hat{B}$ do not commute, then no general conclusion can be drawn about the solvability of (1.37) except for special cases. For example, if $[\hat{A},\ \hat{B}]$ is a non-zero constant then (1.37) evidently does not admit a non-trivial solution. This is the case for the pair of position and momentum operators $\hat{q}$ and $\hat{p}$. Hence, $\hat{q}$ and $\hat{p}$ do not have any common eigenvector. Same result can be shown to hold for angular momentum operators. It is generally accepted that non-commuting observables of common interest do not possess common eigenvectors. However, it can be proved that if $\hat{A}$ and $\hat{B}$ do not commute then they do not admit a common set of eigenvectors [3].

14. If a state $|\psi(t)\rangle$ or an operator $\hat{A}(t)$ is a function of a scalar t then its *derivative* with respect to t has the usual meaning of the calculus of c-number functions. The rules for differentiation of a product of states or of operators with respect to a parameter are also same as for the c-number functions provided that in applying those rules the order of states and operators is retained. Thus, for example,

$$\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle\langle\phi(t)| = \left(\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle\right)\langle\phi(t)| + |\psi(t)|\rangle\frac{\mathrm{d}}{\mathrm{d}t}|\phi(t)\rangle,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\phi(t)|\psi(t)\rangle = \left(\frac{\mathrm{d}}{\mathrm{d}t}\langle\phi(t)\right)|\psi(t)\rangle + \langle\phi(t)|\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle; \tag{1.38}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{A}_1\cdots\hat{A}_n = \left(\frac{\mathrm{d}}{\mathrm{d}t}\hat{A}_1\right)\cdots\hat{A}_n + \cdots + \hat{A}_1\cdots\frac{\mathrm{d}}{\mathrm{d}t}\hat{A}_n. \tag{1.39}$$

Hence, verify that

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\hat{A}^n\right) = \sum_{k=0}^{n-1}\hat{A}^k\left(\frac{\mathrm{d}}{\mathrm{d}t}\hat{A}\right)\hat{A}^{n-k-1}. \tag{1.40}$$

15. Consider the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = \hat{A}(t)|\psi(t)\rangle \tag{1.41}$$

Its solution may be written in the form

$$|\psi(t)\rangle = \overleftarrow{T}\exp\left[\int_{t_0}^{t}\mathrm{d}\tau\hat{A}(\tau)\right]|\psi(t_0)\rangle \tag{1.42}$$

where the *time-ordered exponential integration* is defined by

$$\overleftarrow{T}\exp\left[\int_{t_0}^{t}\mathrm{d}\tau\hat{A}(\tau)\right] = 1 + \int_{t_0}^{t}\mathrm{d}\tau\hat{A}(\tau)$$
$$+\int_{t_0}^{t}\mathrm{d}\tau_2\int_{t_0}^{\tau_2}\mathrm{d}\tau_1\hat{A}(\tau_2)\hat{A}(\tau_1) + \cdots$$
$$+\int_{t_0}^{t}\mathrm{d}\tau_n\int_{t_0}^{\tau_n}\mathrm{d}\tau_{n-1}\cdots\int_{t_0}^{\tau_2}\mathrm{d}\tau_1\hat{A}(\tau_n)\cdots\hat{A}(\tau_1) + \cdots. \tag{1.43}$$

Here $\overleftarrow{T}$ is the so called *time-ordering* operator. It arranges operators in a chronological order with time increasing from right to left. Verify by term-by-term differentiation of (1.43) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\overleftarrow{T}\exp\Big[\int_{t_0}^{t}\mathrm{d}\tau\hat{A}(\tau)\Big]=\hat{A}(t)\overleftarrow{T}\exp\Big[\int_{t_0}^{t}\mathrm{d}\tau\hat{A}(\tau)\Big]. \tag{1.44}$$

On using this it follows that (1.42) indeed satisfies (1.41). In the following we list some properties of the time-ordered exponential operation.

- Take the hermitian conjugate of (1.43) to show that
$$\Big\{\overleftarrow{T}\exp\Big[\int_{t_0}^{t}\mathrm{d}\tau\hat{A}(\tau)\Big]\Big\}^{\dagger}=\overrightarrow{T}\exp\Big[\int_{t_0}^{t}\mathrm{d}\tau\hat{A}^{\dagger}(\tau)\Big]. \tag{1.45}$$
The operator $\overrightarrow{T}$ in the equation above arranges operator in a chronological order with time increasing from left to right. Verify that
$$\frac{\mathrm{d}}{\mathrm{d}t}\overrightarrow{T}\exp\Big[\int_{t_0}^{t}\mathrm{d}\tau\hat{A}(\tau)\Big]=\overrightarrow{T}\exp\Big[\int_{t_0}^{t}\mathrm{d}\tau\hat{A}(\tau)\Big]\hat{A}(t). \tag{1.46}$$
- If $\big[\hat{A}(t_i),\ \hat{A}(t_j)\big]=0$ for all t_i, t_j then the operators under the integral in (1.43) may be shuffled at will like the c-numbers. This property leads to the relation
$$\int_{t_0}^{t}\mathrm{d}\tau_n\int_{t_0}^{\tau_n}\mathrm{d}\tau_{n-1}\cdots\int_{t_0}^{\tau_2}\mathrm{d}\tau_1\hat{A}(\tau_n)\cdots\hat{A}(\tau_1)$$
$$=\frac{1}{n!}\Big[\int_{t_0}^{t}\mathrm{d}\tau\hat{A}(\tau)\Big]^{n}. \tag{1.47}$$
Substitution of this in (1.43) yields
$$\overleftarrow{T}\exp\Big[\int_{t_0}^{t}\mathrm{d}\tau\hat{A}(\tau)\Big]=\exp\Big[\int_{t_0}^{t}\mathrm{d}\tau\hat{A}(\tau)\Big]. \tag{1.48}$$
In particular, if $\hat{A}$ is independent of t then
$$\overleftarrow{T}\exp\Big[\int_{t_0}^{t}\mathrm{d}\tau\hat{A}\Big]=\exp[\hat{A}(t-t_0)]. \tag{1.49}$$
- If $\hat{A}(t)$ does not commute at different times then the commutator of $\hat{A}(t_i)$ and $\hat{A}(t_j)$ would contribute to the integral if $\hat{A}(t_i)$ and $\hat{A}(t_j)$ are interchanged. Hence, on shuffling the operators as in (1.47), we can express the n^{th} term in the time-ordered integral in the form
$$\int_{t_0}^{t}\mathrm{d}\tau_n\int_{t_0}^{\tau_n}\mathrm{d}\tau_{n-1}\cdots\int_{t_0}^{\tau_2}\mathrm{d}\tau_1\hat{A}(\tau_n)\cdots\hat{A}(\tau_1)$$
$$=\frac{1}{n!}\Big[\int_{t_0}^{t}\mathrm{d}\tau\hat{A}(\tau)\Big]^{n}+\hat{C}_n(t), \tag{1.50}$$
the $\hat{C}_n(t)$ being the contribution from commutators of $\hat{A}(t)$ at defferent times. This reduces to (1.47) if the commutator of $\hat{A}(t)$ at different times vanishes.
- If $\hat{A}$ is time-independent but $\hat{B}(t)$ a function of time then
$$\overleftarrow{T}\exp\Big[\int_{t_0}^{t}\mathrm{d}\tau\Big\{\hat{A}+B(\tau)\Big\}\Big]$$

$$= \exp[\hat{A}(t-t_0)]\overleftarrow{T}\exp\left[\int_{t_0}^{t} \mathrm{d}\tau \hat{\tilde{B}}(\tau)\right], \tag{1.51}$$

$$\hat{\tilde{B}}(t) = \exp[-\hat{A}(t-t_0)]\hat{B}\exp[\hat{A}(t-t_0)]. \tag{1.52}$$

The relation (1.51) may be established by showing that the terms on its two sides have the same derivative with respect to t.

The problem of evaluating a time-ordered exponential integral is addressed in Chap. 2.

Superoperators. We have so far considered the operation of linear transformation of vectors in a Hilbert space. Another important class of operations consists of transformation of operators. A *superoperator* defines a rule that associates an operator with another operator acting in the same Hilbert space. The operation of transformation of an operator $\hat{f}$ to another operator $\hat{g}$ may be expressed as

$$\hat{\hat{L}}\hat{f} = \hat{g}. \tag{1.53}$$

Here the superoperator $\hat{\hat{L}}$, distinguished from operators by a double caret, defines the rule of the transformation. We restrict our attention to linear superoperators acting on linear operators in a given Hilbert space.

Recall from the theory of vector spaces that linear operators acting in a vector space constitute a vector space. The relation (1.53) may then be viewed as defining a transformation in a vector space and a superoperator may be identified as an operator in the vector space of the operators. Furthermore, the vector space of operators may be made a scalar product space by defining a suitable scalar product. A useful definition of scalar product is

$$\left(\hat{A},\ \hat{B}\right) = \mathrm{Tr}\left[\hat{A}^{\dagger}\hat{B}\right]. \tag{1.54}$$

It may be verified that this definition is in accordance with the axioms of the scalar product. In analogy with operators, the definition (1.54) of the scalar product leads to the following definition of the *adjoint* $\hat{\hat{L}}^{\dagger}$ of $\hat{\hat{L}}$:

$$\mathrm{Tr}\left[\hat{f}^{\dagger}\hat{\hat{L}}^{\dagger}\hat{g}\right] = \mathrm{Tr}\left[\hat{g}^{\dagger}\hat{\hat{L}}\hat{f}\right]^{*}. \tag{1.55}$$

In order to obtain a c-number representation of superoperators, let $\{|e_i\rangle\}$ be a complete set of orthonormal vectors spanning an N-dimensional Hilbert space. The transformation (1.53) in this basis may be written as

$$g_{ij} = \sum_{k,l=1}^{N} L_{ij,kl} f_{kl}. \tag{1.56}$$

The superoperator $\hat{\hat{L}}$ is thus represented by a tensor $\{L_{ij,kl}\}$ acting on a matrix. If $N \times N$ matrices $\{f_{ij}\}$ and $\{g_{ij}\}$ are represented as column vectors having N^2 elements then $L_{ij,kl}$ is represented as an $N^2 \times N^2$ matrix. This provides a means of converting equations involving superoperators in to matrix equations.

1.1.2 Postulate 2

To each dynamical variable there corresponds a unique hermitian operator.

The reason for associating a hermitian operator with a dynamical variable will become clear after the statement of the postulate 4.

1.1.3 Postulate 3

If $\hat{A}$ and $\hat{B}$ are hermitian operators corresponding to classical dynamical variables a and b then the commutator of $\hat{A}$ and $\hat{B}$ is given by

$$[\hat{A},\ \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A} = \mathrm{i}\hbar\{a, b\}, \tag{1.57}$$

where $\{a, b\}$ is the classical Poisson bracket of a and b and $\hbar = h/2\pi$ where h is the Planck's constant. See [1] for the rationale behind this postulate.

1.1.4 Postulate 4

Each act of measurement of an observable $\hat{A}$ of a system in state $|\psi\rangle$ collapses the system to an eigenstate $|a_i\rangle$ of $\hat{A}$ with probability $|\langle a_i|\psi\rangle|^2$. The average or the expectation value of $\hat{A}$ is given by

$$\langle\hat{A}\rangle = \sum_i a_i |\langle a_i|\psi\rangle|^2 = \langle\psi|\hat{A}|\psi\rangle, \tag{1.58}$$

the a_i being the eigenvalue of $\hat{A}$ corresponding to the eigenstate $|a_i\rangle$.

The complex number $\langle a_i|\psi\rangle$ is called the *probability amplitude*. The last equality in (1.58) can be derived by (i) rewriting $|\langle a_i|\psi\rangle|^2$ as $\langle\psi|a_i\rangle\langle a_i|\psi\rangle$, and (ii) by invoking (1.34) to write the summation over i of $a_i|a_i\rangle\langle a_i|$ as $\hat{A}$.

An operationally useful way of evaluating the probability $|\langle a_i|\psi\rangle|^2$ is to use the expression (10.11) to write

$$|a_i\rangle\langle a_i| = \Big[\Pi^n_{j\neq i}\Big(a_j - a_i\Big)\Big]^{-1} \Pi^n_{j\neq i}\Big(a_j - \hat{A}\Big). \tag{1.59}$$

This assumes that $a_1, \ldots, a_n$ are distinct. As a consequence, the probability of observing the eigenvalue a_i as an outcome of a measurement of $\hat{A}$ on $|\psi\rangle$ is given by

$$|\langle a_i|\psi\rangle|^2 = \Big[\Pi^n_{j\neq i}\Big(a_j - a_i\Big)\Big]^{-1} \Big\langle\psi\Big|\Pi^n_{j\neq i}\Big(a_j - \hat{A}\Big)\Big|\psi\Big\rangle. \tag{1.60}$$

In the discussion above, it is tacitly assumed that the eigenvalues are denumerable. Let the eigenvalues ξ of the operator associated with the observable be continuous with $|\xi\rangle$ as the corresponding eigenvectors. Then, $|\langle\xi|\psi\rangle|^2$ is identified as the *probability density* so that $|\langle\xi|\psi\rangle|^2\,\mathrm{d}\xi$ is the probability that the act of measurement results in a value in the range $\mathrm{d}\xi$ around ξ. Invoke (A.1) to show that

$$|\langle\xi|\psi\rangle|^2 = \langle\psi|\delta(\xi - \hat{A})|\psi\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}x \langle\psi|\exp[\mathrm{i}(\xi - \hat{A})x]|\psi\rangle. \tag{1.61}$$

Since, according to the postulate 2, the observables are represented by linear hermitian operators whose eigenvalues are necessarily real, it follows that the act of measurement would give a real number as its result as any act of measurement does. This also explains the rationale behind associating a hermitian operator with an observable.

In the following we list some implications of this postulate.

- As a consequence of this postulate, quantum theory predicts only the average of the results of measurements on a large number of identically prepared systems. The results of all such measurements are identical only if the observed state is an eigenstate of the measured observable. Only in that case does the observable possess a definite value, that value being the corresponding eigenvalue. Also, as mentioned before, non-commuting observables do not admit common eigenvectors. Hence, non-commuting observables can not have definite values simultaneously. Simultaneous measurement of non-commuting observables to an arbitrary degree of accuracy is thus *incompatible*. Non-commuting observables are *complementary* in the sense that precise knowledge of one excludes that of the other.
- Since quantum predictions are probabilistic, it is important to know the extent of the spread in the outcomes of measurements. A measure of the spread in the values of the results of measurements of an observables $\hat{A}$ of a system in state $|\psi\rangle$ is the *variance* defined by

$$\Delta\hat{A}^2 = \langle\psi|[\hat{A} - \langle\hat{A}\rangle]^2|\psi\rangle = \langle\psi|\hat{A}^2|\psi\rangle - \langle\psi|\hat{A}|\psi\rangle^2. \tag{1.62}$$

In order to determine the relationship between variances of two observables $\hat{A}$ and $\hat{B}$ due to measurements on a system in state $|\psi\rangle$, let

$$|\psi_1\rangle = [\hat{A} - \langle\hat{A}\rangle]|\psi\rangle, \qquad |\psi_2\rangle = [\hat{B} - \langle\hat{B}\rangle]|\psi\rangle. \tag{1.63}$$

Invoke the Schwarz inequality (1.5) to arrive at the relation

$$\Delta\hat{A}^2\Delta\hat{B}^2 \geq \frac{1}{4}\Big[\langle\hat{F}\rangle^2 + \langle\hat{C}\rangle^2\Big] \tag{1.64}$$

called the *uncertainty relation*. Here

$$[\hat{A},\ \hat{B}] = \mathrm{i}\hat{C}, \qquad \hat{F} = \hat{A}\hat{B} + \hat{B}\hat{A} - 2\langle\hat{A}\rangle\langle\hat{B}\rangle. \tag{1.65}$$

Note that, as a result of assumed hermiticity of $\hat{A}$ and $\hat{B}$, the operators $\hat{C}$ and $\hat{F}$ are also hermitian. The operator $\hat{F}$ is a measure of correlations between $\hat{A}$ and $\hat{B}$. The uncertainty product is minimum i.e. the equality in (1.64) holds iff, following (1.6), $|\psi_1\rangle = -\mathrm{i}\lambda|\psi_2\rangle$, where λ is a complex number, i.e. iff

$$[\hat{A} + \mathrm{i}\lambda\hat{B}]|\psi\rangle = \Big[\langle\hat{A}\rangle + \mathrm{i}\lambda\langle\hat{B}\rangle\Big]|\psi\rangle \ \equiv\ z|\psi\rangle. \tag{1.66}$$

The state $|\psi\rangle$ satisfying (1.66) is called a *minimum uncertainty state.* We may derive useful general results about the solvability of (1.66) and the expressions for $\Delta\hat{A}^2$ and $\Delta\hat{B}^2$ in the minimum uncertainty state. To

that end, verify that $\hat{A} + \mathrm{i}\lambda\hat{B}$ is a normal operator if $\mathrm{Re}(\lambda) = 0$. Recall that the eigenstates of only a normal operator are orthonormal. Hence it follows that the minimum uncertainty states (1.66) for a given λ are non-orthogonal if $\mathrm{Re}(\lambda) \neq 0$.

Next, rewrite (1.66) in the form

$$[\hat{A} - \langle\hat{A}\rangle]|\psi\rangle = -\mathrm{i}\lambda[\hat{B} - \langle\hat{B}\rangle]|\psi\rangle. \tag{1.67}$$

Operate this on the left with $\hat{A} - \langle\hat{A}\rangle$ and take the scalar product of the resulting vector with $|\psi\rangle$. Repeat this procedure by operating (1.67) now with $\hat{B} - \langle\hat{B}\rangle$. On using (1.65), the two equations so obtained read

$$\Delta\hat{A}^2 = -\frac{\mathrm{i}\lambda}{2}\Big[\langle\hat{F}\rangle + \mathrm{i}\langle\hat{C}\rangle\Big], \quad \Delta\hat{B}^2 = \frac{\mathrm{i}}{2\lambda}\Big[\langle\hat{F}\rangle - \mathrm{i}\langle\hat{C}\rangle\Big]. \tag{1.68}$$

Set $\lambda = \lambda_\mathrm{r} + \mathrm{i}\lambda_\mathrm{i}$. Compare the real and imaginary parts of each of the equations above to get

$$\Delta\hat{A}^2 = \frac{1}{2}\Big[\lambda_\mathrm{i}\langle\hat{F}\rangle + \lambda_\mathrm{r}\langle\hat{C}\rangle\Big], \quad \Delta\hat{B}^2 = \frac{1}{|\lambda|^2}\Delta\hat{A}^2,$$
$$\lambda_\mathrm{i}\langle\hat{C}\rangle - \lambda_\mathrm{r}\langle\hat{F}\rangle = 0. \tag{1.69}$$

These relations imply that

1. If $|\lambda| = 1$ then $\Delta\hat{A}^2 = \Delta\hat{B}^2$. The corresponding states may then be referred to as *equal variance minimum uncertainty states* .
2. If $|\lambda| = 1$ along with $\lambda_\mathrm{i} = 0$ then $\Delta\hat{A}^2 = \Delta\hat{B}^2$ and $\langle\hat{F}\rangle = 0$. Since $\hat{F}$ is a measure of correlations between $\hat{A}$ and $\hat{B}$, the corresponding states may be referred to as *uncorrelated equal variance minimum uncertainty states.*
3. If $\lambda_\mathrm{r} \neq 0$ then
$$\langle\hat{F}\rangle = \frac{\lambda_\mathrm{i}}{\lambda_\mathrm{r}}\langle\hat{C}\rangle, \quad \Delta\hat{A}^2 = \frac{|\lambda|^2}{2\lambda_\mathrm{r}}\langle\hat{C}\rangle, \quad \Delta\hat{B}^2 = \frac{1}{2\lambda_\mathrm{r}}\langle\hat{C}\rangle. \tag{1.70}$$
The variances and the correlations in the measurement of two observables in this case is expressible in terms of the average of their commutator $\hat{C}$. Clearly, those quantities are completely determined without an explicit construction of the said state if $\hat{C}$ is a constant multiple of the identity operator. Also, since $\Delta\hat{A}^2, \Delta\hat{B}^2$ are positive, it follows that λ_r should have the same sign as $\langle\hat{C}\rangle$. Hence, if $\hat{C}$ is a positive operator then admissible solutions of (1.66) are obtained only if $\lambda_\mathrm{r} > 0$. As an example, recall that $\hat{C} = \hbar I > 0$ for the pair $(\hat{q}, \hat{p})$. Hence, the minimum uncertainty states for the pair $(\hat{q}, \hat{p})$ exist only if $\lambda_\mathrm{r} > 0$. This is borne out also by an explicit solution of (1.66) carried in Sect. 3.2.

1.1.5 Postulate 5

The time evolution of a state $|\psi\rangle$ is governed by the Schrödinger equation

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle, \tag{1.71}$$

where $\hat{H}(t)$ is the Hamiltonian which is a hermitian operator associated with the total energy of the system.

This postulate provides the means for determining the dynamical evolution of a system. The Schrödinger equation (1.71) is of the form (1.41). Its solution is, therefore, given by

$$|\psi(t)\rangle = \overleftarrow{T} \exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t} \mathrm{d}\tau \hat{H}(\tau)\right)|\psi(0)\rangle \equiv \hat{U}_S(t,t_0)|\psi(t_0)\rangle. \tag{1.72}$$

Invoke (1.45) to show that

$$\langle\psi(t)| = \langle\psi(0)|\overrightarrow{T} \exp\left(\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t} \mathrm{d}\tau \hat{H}(\tau)\right) \equiv \langle\psi(0)|\hat{U}_S^{\dagger}(t,t_0). \tag{1.73}$$

If $\hat{H}$ is time-independent then, on using (1.49), (1.72) reads

$$|\psi(t)\rangle = \exp\left(-\mathrm{i}(t-t_0)\hat{H}/\hbar\right)|\psi(t_0)\rangle. \tag{1.74}$$

Expand $|\psi(t_0)\rangle$ in the basis of the eigenstates $|E_i\rangle$ of $\hat{H}$ and apply (1.33) to reduce (1.74) to

$$|\psi(t)\rangle = \sum_{i=1}^{n} \exp\left(-\mathrm{i}E_i(t-t_0)t/\hbar\right)\langle E_i|\psi(t_0)\rangle|E_i\rangle. \tag{1.75}$$

Now, the quantities of physical interest are the expectation values of operators. Using (1.72), the expectation value $\langle\psi(t)|\hat{A}|\psi(t)\rangle$ of $\hat{A}$ at time t may be expressed as

$$\langle\psi(t)|\hat{A}|\psi(t)\rangle = \langle\psi(t_0)|\hat{A}(t)|\psi(t_0)\rangle, \tag{1.76}$$

where

$$\hat{A}(t) = \hat{U}_S^{\dagger}(t,t_0)\hat{A}\hat{U}_S(t,t_0), \tag{1.77}$$

now carries the time dependence. It is straightforward to see that $\hat{A}(t)$ evolves according to the *Heisenberg equation*

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\hat{A}(t) = \left[\hat{A},\ \hat{H}(t)\right]. \tag{1.78}$$

The time evolution of the expectation value of an operator can thus be pictured either in the framework of the time evolution of the state, called the *Schrödinger picture* or in that of the operator, called the *Heisenberg picture.*

Next, consider a system described by $|\psi(t)\rangle$ evolving under the action of a hamiltonian $\hat{H}$ decomposable as

$$\hat{H} = \hat{H}_0 + \hat{H}_1(t) \tag{1.79}$$

where $\hat{H}_0$ is time-independent. Define

$$|\psi_I(t)\rangle = \exp\left(\mathrm{i}\hat{H}_0 t/\hbar\right)|\psi(t)\rangle. \tag{1.80}$$

It is then straightforward to see that $|\psi_I(t)\rangle$ evolves according to

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi_I(t)\rangle = \hat{H}_I(t)|\psi_I(t)\rangle, \tag{1.81}$$

$$\hat{H}_I(t) = \exp\left(\mathrm{i}\hat{H}_0 t/\hbar\right)\hat{H}_1(t)\exp\left(-\mathrm{i}\hat{H}_0 t/\hbar\right). \tag{1.82}$$

The evolution (1.81) is said to be in the *interaction picture* generated by $\hat{H}_0$.

In many situations, it is required to know the *multi time joint probability* $p(\{|\phi_i\rangle, t_i\})$ that a system in a state $|\phi_0(t_0)\rangle$ at t_0 is found in the state $|\phi_i\rangle$ at $t = t_i$ $(i = 1, \ldots, n)$. To find that probability, note from (1.72) that the state of the system at time t_1 is given by $\hat{U}_S(t_1, t_0)|\phi_0(t_0)\rangle$ and its projection on $|\phi_1\rangle$ is $|\phi_1(t_1)\rangle = |\phi_1\rangle\langle\phi_1|\hat{U}_S(t_1, t_0)|\phi_0(t_0)\rangle$. The state $|\phi_1(t_1)\rangle$ then evolves till time t_2 to $\hat{U}_S(t_2, t_1)|\phi_1(t_1)\rangle$ whose projection along $|\phi_2\rangle$ is $|\phi_2(t_2)\rangle = |\phi_2\rangle\langle\phi_2|\hat{U}_S(t_2, t_1)|\phi_1(t_1)\rangle$. On continuing this argument till time t_n it follows that

$$p(\{|\phi_i\rangle, t_i\}) = \Big|\prod_{i=1}^{n}\langle\phi_i|U_S(t_i, t_{i-1})|\phi_{i-1}\rangle\Big|^2. \tag{1.83}$$

Consider now a time-independent hamiltonian so that, by virtue of (1.74), $\hat{U}_S(t_i, t_j) = \exp(-\mathrm{i}\hat{H}(t_i - t_j)/\hbar)$. Also, let the observations be spaced at equal time intervals $t_i - t_{i-1} = t/n$. The probability that at each time t_i the system is observed in its initial state $|\phi_0\rangle$ then reads

$$p(\{|\phi_0\rangle, t_i\}) = \Big|\langle\phi_0|\exp\Big(-\mathrm{i}\hat{H}t/n\hbar\Big)|\phi_0\rangle\Big|^{2n}. \tag{1.84}$$

Let $t/n \ll 1$. Expand the exponential in (1.84) and retain terms up to t^2 to arrive at

$$\Big|\langle\phi_0|\exp\Big(-\mathrm{i}\hat{H}t/n\hbar\Big)|\phi_0\rangle\Big|^2 \approx 1 - \Big(\frac{t}{n\hbar}\Big)^2\Delta H^2, \tag{1.85}$$

where $\Delta\hat{H}^2 = \langle\phi_0|\hat{H}^2|\phi_0\rangle - \langle\phi_0|\hat{H}|\phi_0\rangle^2$. The joint probability that the system is observed in its initial state at the time of each of n equally spaced observations on it in time t is then given, on substituting (1.85) in (1.84), by [4]

$$p(\{|\phi_0\rangle, t_i\}) = \Big[1 - \Big(\frac{t}{\hbar n}\Big)^2\Delta\hat{H}^2\Big]^n. \tag{1.86}$$

Compare this with the probability $p(|\phi_0\rangle, t)$ that the system is found in the initial state at time t if it is left unobserved in between. That probability is evidently (1.86) corresponding to $n = 1$, i.e.

$$p(|\phi_0\rangle, t) = 1 - t^2\Delta\hat{H}^2/\hbar^2. \tag{1.87}$$

This shows that the probability of finding the system in its initial state at a given time is increased if it is observed repeatedly at intermediate times compared with that probability when the system is left unobserved during its evolution to that time. In fact, for $n \gg 1$, (1.86) approaches $\exp(-t^2\Delta H^2/n\hbar^2)$. Hence, the probability of finding the system in its initial state tends to unity

as the number n of observations tends to infinity. In other words, a system continuously under observation does not evolve! This is the *quantum Zeno effect* or the *watchdog effect*. This effect was invoked to predict the inhibition of decay of an unstable system [5]. An experimental demonstration of this effect in the context of quantum optics has been reported in [6]. For a discussion of various interpretations of the experimental results, see [7].

1.2 Geometric Phase

Yet another characteristic which exhibits distinctly quantum nature of evolution is the phase of a state. The observable effects of phase obviously can not be exhibited by the expectation value $\langle\psi|\hat{A}|\psi\rangle$ of an observable $\hat{A}$ as it is independent of the phase of $|\psi\rangle$. Effects of phase may be manifested, as we will see in Sect. 1.7, in interference between two non-orthogonal states.

Now, consider a state $|\psi(t)\rangle$ evolving under the action of a Hamiltonian $\hat{H}(t)$ according to the Schrödinger equation. The state is assumed to be normalized to unity. The phase difference between the states at two times t_1 and t_2, called the *total phase*, is given by

$$\phi_{\mathrm{t}} = \arg\langle\psi(t_1)|\psi(t_2)\rangle. \tag{1.88}$$

The phase is defined modulo 2π. It turns out to be useful to introduce another phase, called *dynamic phase* , defined by

$$\phi_{\mathrm{d}} = -\frac{1}{\hbar}\int_{t_1}^{t_2}\langle\psi(\tau)|\hat{H}(\tau)|\psi(\tau)\rangle\,\mathrm{d}\tau = \mathrm{Im}\int_{t_1}^{t_2}\langle\psi(\tau)|\dot{\psi}(\tau)\rangle\mathrm{d}\tau. \tag{1.89}$$

The last equality above is the result of the assumption that $|\psi(t)\rangle$ evolves according to the Schrödinger equation and that $\langle\psi|\hat{H}|\psi\rangle$ is real. If $\hat{H}$ is time-independent, and if the initial state is an eigenstate of $\hat{H}$ with eigenvalue E_n then, invoke (1.72) to show that $\phi_{\mathrm{t}} = \phi_{\mathrm{d}} = E_n(t_1 - t_2)/\hbar$ i.e. the dynamic phase in this case is the same as the total phase. In general, the difference

$$\phi_{\mathrm{g}} = \phi_{\mathrm{t}} - \phi_{\mathrm{d}} = \arg\langle\psi(t_1)|\psi(t_2)\rangle - \mathrm{Im}\int_{t_1}^{t_2}\mathrm{d}\tau\langle\psi(\tau)|\dot{\psi}(\tau)\rangle \tag{1.90}$$

is called, for the reason elaborated below, the *geometric phase*. We note first that the definition (1.90) holds even for the states which do not satisfy the Schrödinger equation. In order to see that, consider the Gauge transformation

$$|\tilde{\psi}(t)\rangle = \exp(\mathrm{i}f(t))|\psi(t)\rangle \tag{1.91}$$

where $f(t)$ is a real smooth function of t. On substituting this for $|\psi(t)\rangle$ in (1.90) we obtain

$$\phi_{\mathrm{g}} = \arg\langle\tilde{\psi}(t_1)|\tilde{\psi}(t_2)\rangle - \mathrm{Im}\int_{t_1}^{t_2}\mathrm{d}\tau\langle\tilde{\psi}(\tau)|\dot{\tilde{\psi}}(\tau)\rangle. \tag{1.92}$$

This shows that ϕ_g remains invariant under the transformation (1.91). Note that, if $|\psi\rangle$ satisfies the Schrödinger equation, $|\tilde{\psi}\rangle$ need not. Now, the transformation (1.91) defines an equivalence class of vectors. Let P be the space obtained by projecting to the same point the states related by (1.91). With the passage of time, any state $|\psi(t)\rangle$ traverses a curve C under the action of $\hat{H}$. Let C_0 be the image of that curve in the projective space P. Let C' be another trajectory traversed by the state vector under the action of $\hat{H}'$. If C' projects on to the same C_0 in the projective space then its geometric phase is clearly the same as that for the motion on C. It can also be verified that ϕ_g is reparameterization invariant, i.e. it is unchanged under the transformation $t \to t'$ where t' is a smooth function of t [8]. The phase ϕ_g is thus a geometric property of unparameterized C_0 in the projective space.

The freedom in the choice of $f(t)$ may be used to cast ϕ_g in different instructive forms. To that end, use (1.91) in the first term on the right hand side of (1.92) so that

$$\phi_g = \arg\langle\psi(t_1)|\psi(t_2)\rangle + f(t_2) - f(t_1) - \mathrm{Im}\int_{t_1}^{t_2} \mathrm{d}\tau\langle\tilde{\psi}(\tau)|\dot{\tilde{\psi}}(\tau)\rangle. \tag{1.93}$$

If $f(t)$ is chosen such that $f(t_1) - f(t_2) = \arg(\langle\psi(t_1)|\psi(t_2)\rangle$ then the geometric phase is the same as the dynamic phase:

$$\phi_g = -\mathrm{Im}\int_{t_1}^{t_2} \mathrm{d}\tau\langle\tilde{\psi}(\tau)|\dot{\tilde{\psi}}(\tau)\rangle = -\phi_d. \tag{1.94}$$

Alternatively, use (1.91) in the second term on the right hand side of (1.92) to get

$$\phi_g = \arg\langle\tilde{\psi}(t_1)|\tilde{\psi}(t_2)\rangle + f(t_1) - f(t_2) - \mathrm{Im}\int_{t_1}^{t_2} \mathrm{d}\tau\langle\psi(\tau)|\dot{\psi}(\tau)\rangle. \tag{1.95}$$

If $f(t)$ is such that

$$f(t_1) - f(t_2) = \mathrm{Im}\int_{t_1}^{t_2} \mathrm{d}\tau\langle\psi(\tau)|\dot{\psi}(\tau)\rangle \tag{1.96}$$

then the geometric phase is the same as the total phase:

$$\phi_g = \arg\langle\tilde{\psi}(t_1)|\tilde{\psi}(t_2)\rangle. \tag{1.97}$$

The geometric phase signifies interesting differential-geometric properties of evolution. For a detailed discussion of its differential geometric interpretation, its generalization to non-hamiltonian evolution and references to experiments on its observation, see [8, 9].

For the sake of illustration, we evaluate next the geometric phase for the evolution generated by the hamiltonian of a harmonic oscillator and that of a two-level system.

1.2.1 Geometric Phase of a Harmonic Oscillator

Recall that the eigenstates of the hamiltonian of a harmonic oscillator of frequency ω are $|n\rangle$ with $E_n = (n+1/2)\hbar\omega$, $(n = 0, 1, \ldots)$ as the corresponding eigenvalues. On invoking (1.75), its state at time t is given by

$$|\psi(t)\rangle = \sum_{m=0}^{\infty} C_m \exp\left(-\mathrm{i}\omega t\left(m + \frac{1}{2}\right)\right)|m\rangle. \tag{1.98}$$

Consider the evolution over the period $2\pi/\omega$. On substituting (1.98) in (1.90) and on using the orthonormality of the states $\{|m\rangle\}$ follows the well-known result

$$\phi_{\mathrm{g}} = 2\pi \sum_{m=0}^{\infty} m|C_m|^2 \equiv 2\pi\langle m\rangle. \tag{1.99}$$

1.2.2 Geometric Phase of a Two-Level System

Consider a system having two states $|\pm\rangle$ in an external field. The state of such a system at any time is a linear combination of $|\pm\rangle$ and hence is expressible as

$$|\psi(t)\rangle = \cos\left(\frac{\theta(t)}{2}\right)|+\rangle + \exp(\mathrm{i}\phi(t))\sin\left(\frac{\theta(t)}{2}\right)|-\rangle \tag{1.100}$$

where the functional form of $\theta(t), \phi(t)$ is unimportant for the present. For the sake of simplicity, assume that θ is independent of time. On substituting (1.100) in (1.90) and after a little algebra it may be shown that [8]

$$\phi_{\mathrm{g}} = \frac{1}{2}\cos(\theta)\,\Delta\phi - \tan^{-1}\left[\cos(\theta)\tan\left(\frac{\Delta\phi}{2}\right)\right] \tag{1.101}$$

with $\Delta\phi = \phi(t_2) - \phi(t_1)$.

1.2.3 Geometric Phase in Adiabatic Evolution

The recent surge in interest in the geometric phase owes its origin to the paper by Berry on the geometric phase in adiabatic evolution [10]. Consider a system evolving under the action of a time-dependent hamiltonian $\hat{H}(t)$. Let the system be initially in an eigenstate $|n(0)\rangle$ of $\hat{H}(0)$. The assumption of adiabadicity means that the state of the system at any time t is

$$|\psi(t)\rangle = \exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^t \mathrm{d}\tau E_n(\tau)\right]|n(t)\rangle \tag{1.102}$$

where $|n(t)\rangle$ is the eigenstate of $\hat{H}(t)$ and $E_n(t)$ the corresponding eigenvalue. Assume also that after a time T the hamiltonian returns to its form at $t = 0$. On substituting (1.102) in (1.90) with $t_1 = 0$, $t_2 = T$ and $|n(0)\rangle = |n(t)\rangle$ it follows that

$$\phi_{\mathrm{g}}^{(n)} = -\mathrm{Im}\int_0^T \langle n(t)|\dot{n}(t)\rangle \mathrm{d}t. \tag{1.103}$$

Now, if it is assumed that the time-dependence of the hamiltonian, and consequently of the states $|n(t)\rangle$, is due to that of the parameters $\{R_i\}$ then (1.103) reduces to the form

$$\phi_{\mathrm{g}}^{(n)} = -\mathrm{Im}\int_0^T \langle n(t)|\nabla_R n(R(t))\rangle.\dot{\boldsymbol{R}}(t) \tag{1.104}$$

which is familiar since its introduction in [10].

1.3 Time-Dependent Approximation Method

We have seen that the problem of studying the time-evolution of a quantum hamiltonian system reduces to solving the Schrödinger equation. However, more often than not, the Schrödinger equation is not exactly solvable. That necessitates use of approximation methods for its solution. The approximation methods are many a times useful to unveil the salient features of even an exactly solvable problem. The approximation method to be employed depends, of course, on the nature of the problem. Here we outline a method of frequent use in quantum optics.

Consider a system whose hamiltonian is expressible as in (1.79) where $\hat{H}_0$ is time-independent whereas $\hat{H}_1(t)$ is time-dependent. Assume that no part of $\hat{H}_1(t)$ commutes with $\hat{H}_0$. In the interaction picture generated by $\hat{H}_0$, the system is described by the state vector $|\psi_I(t)\rangle$ related to the state vector $|\psi(t)\rangle$ in the Schrödinger picture by (1.80). The formal solution of (1.81) governing its evolution is

$$|\psi_I(t)\rangle = \hat{U}_I(t)|\psi(0)\rangle, \quad \hat{U}_I(t) = \overleftarrow{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}\tau \hat{H}_I(\tau)/\hbar\right), \tag{1.105}$$

where $\hat{H}_I(t)$ is defined in (1.82). Let $\hat{H}_0$ and $\hat{H}_1(t)$ be such that

$$\hat{H}_I(t) = \hbar\sum_{k=1}^{N}\left[\hat{F}_k \exp\left(-\mathrm{i}\Omega_k\ t\right) + \hat{F}_k^{\dagger}\exp\left(\mathrm{i}\Omega_k\ t\right)\right]. \tag{1.106}$$

Substitute this in $\hat{U}_I(t)$ of (1.105). Expand $\hat{U}_I(t)$ as in (1.43) and express its n^{th} term as in (1.50). It follows that if

$$||\hat{F}_{\boldsymbol{k}}||/\Omega_{\boldsymbol{k}} \ll 1 \tag{1.107}$$

then the time-ordered expansion of $\hat{U}_I(t)$ is a perturbative expansion in the smallness parameter $||\hat{F}_{\boldsymbol{k}}||/\Omega_{\boldsymbol{k}}$. It may be terminated at a desired order.

A more instructive form of perturbation expansion is obtained by separating the contribution from commutators of $\hat{H}_I(t)$ at different times. To that end, (i) express the time-ordered expansion of $\hat{U}_I(t)$ in the form

$\hat{U}_I(t) = 1 + \hat{x} = \exp(\ln(1+\hat{x}))$, (ii) expand $(1+\hat{x})$ in powers of $\hat{x}$, (iii) group together the terms having the same number of $\hat{H}_I(t)'s$ to get

$$\hat{U}_I(t) = \exp\left[\sum_{k=1}^{\infty} \hat{M}_k(t)\right], \tag{1.108}$$

$$\hat{M}_1(t) = -\frac{\mathrm{i}}{\hbar}\int_0^t \mathrm{d}\tau \hat{H}_I(\tau),$$
$$\hat{M}_2(t) = \left(-\frac{\mathrm{i}}{\hbar}\right)^2 \int_0^t \mathrm{d}\tau_2 \int_0^{\tau_2} \mathrm{d}\tau_1 \hat{H}_I(\tau_2)\hat{H}_I(\tau_1) - \frac{1}{2}\hat{M}_1^2(t) \tag{1.109}$$

and so on. Comparison of (1.108) and (1.50) shows that if $\hat{H}_I(t)$ commutes at any two times then $\hat{M}_k(t) = 0$ for $k \geq 2$. Hence, $\hat{M}_k(t)$ for $k \geq 2$ contain contribution from the commutators of $\hat{H}_I(t)$ at different times.

1.4 Quantum Mechanics of a Composite System

The state vector provides a quantum theoretic description of an isolated system. At times we need to describe the state of a system in terms of its constituents known as its *subsystems*. Here we outline the approach for the quantum theoretic description of a system in terms of its subsystems.

For the sake of simplicity, consider a system made up of two subsystems: A and B. The state vector $|\Psi_{A+B}\rangle$ of the combined system may then be expressed in terms of orthonormal basis vectors $\{|a_i\rangle\}$ and $\{|b_i\rangle\}$ respectively for the subsystems A and B as

$$|\Psi_{A+B}\rangle = \sum_{i,j} \alpha_{ij}|a_i, b_j\rangle, \quad \alpha_{ij} = \langle a_i, b_j|\Psi_{A+B}\rangle. \tag{1.110}$$

where $\{|a_i, b_j\rangle\} \equiv \{|a_i\rangle\} \otimes \{|b_j\rangle\}$ is the *direct product* of the sets of vectors $\{|a_i\rangle\}$ and $\{|b_i\rangle\}$. Let $|\Psi_{A+B}\rangle$ be normalized to unity so that

$$\sum_{i,j=1} |\alpha_{ij}|^2 = 1. \tag{1.111}$$

Now, if $\alpha_{ij} = \alpha_i^{(A)}\alpha_j^{(B)}$ then (1.110) shows that

$$|\Psi_{A+B}\rangle = \left[\sum_i \alpha_i^{(A)}|a_i\rangle\right]\left[\sum_j \alpha_i^{(B)}|b_j\rangle\right] \equiv |\psi_A\rangle|\psi_B\rangle, \tag{1.112}$$

where $|\psi_A\rangle$ ($|\psi_B\rangle$) is the state vector only of the subsystem A (B). In this case the state vector of the combined system factorizes in to those of its subsystems. The state of a composite system which can not be factorized in to a product of the states of its subsystems is called an *entangled state*. The entangled states play a crucial role in understanding purely quantum effects.

1.5 Quantum Mechanics of a Subsystem and Density Operator

Consider a system composed of subsystems A and B. Let it be that we are interested in the behaviour of only one of the subsystems, say, the subsystem A. That behaviour is determined by the expectation values of the operators $\{\hat{X}_A\}$ which act on the states of the subsystem A alone. On using (1.110) for the state vector of the composite system, the expectation value of $\hat{X}_A$ is seen to be given by

$$\begin{aligned}\langle \hat{X}_A \rangle &= \langle \Psi_{A+B} | \hat{X}_A | \Psi_{A+B} \rangle = \sum_{k,l} \sum_{i,j} \alpha^*_{kl} \alpha_{ij} \langle a_k | \hat{X}_A | a_i \rangle \langle b_l | b_j \rangle \\ &= \sum_{i,j,k} \alpha^*_{kj} \alpha_{ij} \langle a_k | \hat{X}_A | a_i \rangle \quad = \sum_{i,k} c_{ik} \langle a_k | \hat{X}_A | a_i \rangle, \end{aligned} \tag{1.113}$$

$$c_{ik} = \sum_j \alpha_{ij} \alpha^*_{kj} = c^*_{ki}. \tag{1.114}$$

In writing the first line in (1.113) we have made use of the fact that $\hat{X}_A$ does not act on the vectors representing the subsystem B, and in writing the second line we have invoked the orthonormality of the basis $\{|b_i\rangle\}$. Use of (1.111) in (1.114) yields

$$\sum_{i=1} c_{ii} = 1. \tag{1.115}$$

Now, on invoking (1.30), (1.113) may be written as

$$\langle \hat{X}_A \rangle = \mathrm{Tr}\left[\hat{X}_A \hat{\rho}_A\right] \tag{1.116}$$

where

$$\hat{\rho}_A = \sum_{i,k} c_{ik} |a_i\rangle \langle a_k|, \tag{1.117}$$

is called the *density operator* of the system A. On taking the matrix element of (1.117) it follows that

$$c_{ik} = \langle a_i | \hat{\rho}_A | a_k \rangle. \tag{1.118}$$

The elements $\{\langle a_i | \hat{\rho}_A | a_j \rangle\}$ constitute a matrix representation of the density operator $\hat{\rho}_A$. Since it is the expectation values of operators which are the quantities of physical interest and since all such expectation values for a subsystem can be found by using the density operator by means of the relation (1.116), it follows that a density operator describes the state of a system interacting with other systems in the same way as a state vector describes the state of an isolated system.

In the following we enumerate some properties of the density operator.

1. The expectation value of an operator $\hat{X}$ of an isolated system in state $|\Psi\rangle$ may be written, using (1.30), as $\langle\Psi|\hat{X}|\Psi\rangle \equiv \mathrm{Tr}[\hat{X}|\Psi\rangle\langle\Psi|]$. On comparing this with (1.116) we see that the density operator of a system in the state $|\Psi\rangle$ is given by $\hat{\rho} = |\Psi\rangle\langle\Psi|$. If the density operator $\hat{\rho}$ of a system is expressible as $\hat{\rho} = |\Psi\rangle\langle\Psi|$ then it is said to be in a *pure state*. Else it is said to be in a *mixed state*.
2. Let Tr_B denote the operation of trace only over the subsystem B. Check, using (1.29), that

$$\mathrm{Tr}_B\Big[|a_i, b_j\rangle\langle a_k, b_l|\Big] = |a_i\rangle\langle a_k|\mathrm{Tr}_B\Big[|b_j\rangle\langle b_l|\Big] = |a_i\rangle\langle a_k|\delta_{jl}. \tag{1.119}$$

Using this to carry the operation of trace over B in $|\Psi_{A+B}\rangle\langle\Psi_{A+B}|$ with $|\Psi_{A+B}\rangle$ given by (1.110) and on invoking (1.114) it follows that

$$\mathrm{Tr}_B\Big[|\Psi_{A+B}\rangle\langle\Psi_{A+B}|\Big] = \mathrm{Tr}_B\left[\hat{\rho}_{A+B}\right] = \hat{\rho}_A. \tag{1.120}$$

3. If the evolution of the system A is governed by the hamiltonian $\hat{H}_A(t)$ then the kets and bras of A evolve according to (1.72) and (1.73). Hence, the density operator (1.117) of A at time t is given by

$$\hat{\rho}_A(t) = \overleftarrow{T}\exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\mathrm{d}\tau\hat{H}_A(\tau)\right]\hat{\rho}_A(0)\overrightarrow{T}\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\mathrm{d}\tau\hat{H}_A(\tau)\right]. \tag{1.121}$$

It is straightforward to verify that the equation of evolution of $\hat{\rho}_A(t)$ is

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}_A(t) = \Big[\hat{H}_A(t),\ \hat{\rho}_A(t)\Big]. \tag{1.122}$$

4. On combining (1.114) and (1.117) we infer that $\hat{\rho}_A = \hat{\rho}_A^\dagger$ i.e. $\hat{\rho}$ is hermitian.
5. The operation of trace over (1.117) combined with (1.115) leads to

$$\mathrm{Tr}(\hat{\rho}_A) = \sum_i c_{ii} = 1. \tag{1.123}$$

6. The probability that a system is in state $|\psi_A\rangle$ is given by the expectation value of $|\psi_A\rangle\langle\psi_A|$ which, by virtue of (1.116), is $\langle\psi_A|\hat{\rho}|\psi_A\rangle$. Since measurable probability should be a positive number, it follows that $\langle\psi_A|\hat{\rho}_A|\psi_A\rangle \geq 0$. Hence $\hat{\rho}_A$ is a positive operator.
7. As a consequence of (4), we note that the eigenvalues of $\hat{\rho}_A$ are real. If $\{|\lambda_i\rangle\}$ are the eigenstates of $\hat{\rho}_A$ corresponding to eigenvalues $\{\lambda_i\}$ then, on applying (1.34), $\hat{\rho}_A$ can be represented as

$$\hat{\rho}_A = \sum_i \lambda_i|\lambda_i\rangle\langle\lambda_i|. \tag{1.124}$$

Since $\hat{\rho}_A \geq 0$, it follows that $\lambda_i \geq 0$. The equation (1.124) also shows that the trace of $\hat{\rho}_A$ is sum of its eigenvalues. This, along with the condition (1.123) imply that $\lambda_i \leq 1$. If one of the $\lambda_i's$, say, $\lambda_1 = 1$ then $\lambda_i = 0$ for all $i \neq 1$. Then, $\hat{\rho}_A = |\lambda_1\rangle\langle\lambda_1|$ which is the density operator for the system in pure state $|\lambda_1\rangle$. On squaring (1.124) and by using the orthonormality of the states $\{|\lambda_i\rangle\}$, it follows that

$$\hat{\rho}_A^2 = \sum_{i=1} \lambda_i^2 |\lambda_i\rangle\langle\lambda_i| \leq \hat{\rho}_A. \tag{1.125}$$

The equality in this holds, as discussed above, only if ρ_A describes a pure state. Since $\mathrm{Tr}[\hat{\rho}_A] = 1$, we note that $\mathrm{Tr}[\hat{\rho}_A^2] = 1$ if the state is pure and that $\mathrm{Tr}[\hat{\rho}_A^2] < 1$ if the state is mixed. Recall from the list of properties of the trace that if $\hat{U}$ is unitary then $\mathrm{Tr}[\hat{A}] = \mathrm{Tr}[\hat{U}\hat{A}\hat{U}^\dagger]$ for any $\hat{A}$. Hence, if $\hat{\rho}$ obeys the inequalty $\mathrm{Tr}[\hat{\rho}^2] < 1$ then so does $\hat{U}\hat{\rho}\hat{U}^\dagger$ if $\hat{U}$ is unitary. A mixed state thus remains mixed and a pure state remains pure under a unitary transformation. Note in particular that the transformation generated by the Schrödinger evolution is unitary. Hence, under the Schrödinger evolution, a mixed state evolves to a mixed state and a pure state to a pure one.

Next we discuss the quantum mechanics of one and two two-state systems which are of interest in the discussion to follow.

1.6 Systems of One and Two Spin-1/2s

Assuming familiarity with the concept of spin in quantum mechanics, we let the vector operator $\hat{\boldsymbol{S}}$ represent spin in the ordinary three-dimensional space. Its components $\hat{S}_x, \hat{S}_y, \hat{S}_z$ in three orthogonal directions, say, the directions $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$, obey the commutation relation (letting $\hbar = 1$ for convenience)

$$\left[\hat{S}_x,\ \hat{S}_y\right] = \mathrm{i}\hat{S}_z, \qquad \left[\hat{S}_z,\ \hat{S}_x\right] = \mathrm{i}\hat{S}_y, \qquad \left[\hat{S}_y,\ \hat{S}_z\right] = \mathrm{i}\hat{S}_x. \tag{1.126}$$

It would turn out to be useful to introduce the *spin raising and lowering operators*

$$\hat{S}_\pm = \hat{S}_x \pm \mathrm{i}\hat{S}_y, \quad \hat{S}_+ = \hat{S}_-^\dagger. \tag{1.127}$$

On applying (1.126), it is straightforward to verify that

$$\left[\hat{S}_+,\ \hat{S}_-\right] = 2\hat{S}_z, \qquad \left[\hat{S}_z,\ \hat{S}_\pm\right] = \pm\hat{S}_\pm. \tag{1.128}$$

It is a spin-1/2 if measurement on any of its components yields one of the two values, $\pm 1/2$. In addition to the commutation relations (1.126), the orthogonal components of a spin-1/2 obey also the anticommutation relations

$$\hat{S}_\mu\hat{S}_\nu + \hat{S}_\nu\hat{S}_\mu = 0,\ \mu \neq \nu = x, y, z. \tag{1.129}$$

The anticommutation relation between the components of spins in two arbitrary directions is obtained by expressing them in terms of the x, y, z components. Verify that

$$\hat{S}_a \hat{S}_b + \hat{S}_b \hat{S}_a = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{2}, \tag{1.130}$$

the $\hat{S}_a$ and $\hat{S}_b$ being the spin components along the directions $\boldsymbol{a}$ and $\boldsymbol{b}$.

By combining the commutation and anti-commutation relations, any product of spin-1/2 operators may be expressed in terms of a single spin-1/2 operator. Thus, for example,

$$\hat{S}_x \hat{S}_y = \frac{\mathrm{i}}{2} \hat{S}_z, \ \hat{S}_y \hat{S}_z = \frac{\mathrm{i}}{2} \hat{S}_x, \ \hat{S}_z \hat{S}_x = \frac{\mathrm{i}}{2} \hat{S}_y \tag{1.131}$$

Express $\hat{S}_\pm$ in terms of $\hat{S}_x$ and $\hat{S}_y$, (see (1.127)), apply the anticommutation relation (1.129) to show that

$$\hat{S}_+ \hat{S}_- + \hat{S}_- \hat{S}_+ = 1. \tag{1.132}$$

By combining (1.132) and (1.128) we find that, for spin-1/2,

$$\hat{S}_+ \hat{S}_- = \frac{1}{2} + \hat{S}_z, \quad \hat{S}_- \hat{S}_+ = \frac{1}{2} - \hat{S}_z. \tag{1.133}$$

Now, if $|\pm\rangle$ are the eigenstates of $\hat{S}_z$ corresponding to the eigenvalues $\pm 1/2$ then verify that

$$\begin{aligned} &\hat{S}_+ = |+\rangle\langle -|, \qquad \hat{S}_- = |-\rangle\langle +|, \\ &\hat{S}_z = \frac{1}{2}\Big(|+\rangle\langle +| - |-\rangle\langle -|\Big), \end{aligned} \tag{1.134a}$$

$$\begin{aligned} &\hat{S}_+|-\rangle = |+\rangle, \quad \hat{S}_-|+\rangle = |-\rangle, \\ &\hat{S}_+|+\rangle = 0, \quad \hat{S}_-|-\rangle = 0, \ \hat{S}_\pm^2 = 0. \end{aligned}$$

$$\hat{S}_+ \hat{S}_z = -\frac{1}{2} \hat{S}_+, \qquad \hat{S}_z \hat{S}_+ = \frac{1}{2} \hat{S}_+ \tag{1.134b}$$

Next, let $\hat{S}_e = \boldsymbol{e}.\hat{\boldsymbol{S}}$ be the operator corresponding to the spin component in direction $\boldsymbol{e}$. Let $p(\pm, \boldsymbol{e})$ be the probability that the outcome of measurement of the component in the direction $\boldsymbol{e}$ of a spin-1/2 in state $|\psi\rangle$ is $\pm 1/2$. On applying (1.60) it is straightforward to show that

$$p(\pm, \boldsymbol{e}) = |\langle \pm | \psi \rangle|^2 = \left\langle \psi \middle| \frac{1}{2} \pm \hat{S}_e \middle| \psi \right\rangle. \tag{1.135}$$

Consider next a system consisting of two spin-1/2 subsystems. Let the eigenvectors $|\pm \boldsymbol{a}, 1\rangle$ of the component $\hat{S}_a^{(1)}$ of spin 1 in direction $\boldsymbol{a}$ be the basis for the Hilbert space of that spin and let the eigenvectors $|\pm \boldsymbol{b}, 2\rangle$ of the component $\hat{S}_b^{(2)}$ of the second spin in direction $\boldsymbol{b}$ constitute the basis vectors for the states of the second spin. A state of a system consisting of these two spins is then a linear combination of the states $|\pm \boldsymbol{a}, \pm \boldsymbol{b}\rangle \equiv |\pm \boldsymbol{a}, 1\rangle \otimes |\pm \boldsymbol{b}, 2\rangle$. This

linear combination can be decomposed to express any state of two spin-1/2s in the form

$$|\psi(\alpha)\rangle = \cos(\alpha)|\boldsymbol{a}_1, -\boldsymbol{a}_2\rangle - \sin(\alpha)|-\boldsymbol{a}_1, \boldsymbol{a}_2\rangle, \quad 0 \le \alpha \le \frac{\pi}{4}. \tag{1.136}$$

This is known as the *Schmidt decomposition* [12] . This state reduces to a product of the states for the two spins for $\alpha = 0$ but is an entangled state for $\alpha \neq 0$. The extent of entanglement may be measured by the absolute value of the correlation function

$$C(\boldsymbol{a}_1, \boldsymbol{a}_2) = \langle \hat{S}^{(1)}_{a_1} \hat{S}^{(2)}_{a_2} \rangle - \langle \hat{S}^{(1)}_{a_1} \rangle \langle \hat{S}^{(2)}_{a_2} \rangle. \tag{1.137}$$

For the state (1.136), it can be verified that

$$|C(\boldsymbol{a}_1, \boldsymbol{a}_2)| = \sin^2(2\alpha)/4. \tag{1.138}$$

This shows that the maximal entanglement is achieved for $\alpha = \pi/4$. In what follows, we consider the maximally entangled singlet state corresponding to $\boldsymbol{a}_1 = \boldsymbol{a}_2 = \boldsymbol{z}$ (say) and $\alpha = \pi/4$ in (1.136):

$$|0\rangle = \sqrt{\frac{1}{2}}\Big[|+,-\rangle - |-,+\rangle\Big]. \tag{1.139}$$

where $|\pm,\pm\rangle$ is the eigenstate of $\hat{S}^{(1)}_z$ and $\hat{S}^{(2)}_z$. The expectation value of $\hat{S}^{(i)}_z$ in this state is straightforwardly zero. By expressing $\hat{S}^{(i)}_{x,y}$ in terms of $\hat{S}_\pm$ using (1.127) and on applying (1.134b), we find that

$$\begin{aligned}
&\langle \hat{S}^{(i)}_x \rangle = \langle \hat{S}^{(i)}_y \rangle = \langle \hat{S}^{(i)}_z \rangle = 0, \\
&\langle \hat{S}^{(i)}_+ \hat{S}^{(j)}_+ \rangle = \langle \hat{S}^{(i)}_- \hat{S}^{(j)}_- \rangle = 0 \\
&\langle \hat{S}^{(i)}_+ \hat{S}^{(j)}_- \rangle = -\frac{1}{2}, \qquad i \neq j
\end{aligned} \tag{1.140}$$

Express the vectors in terms of their Cartesian components and use these results to show that, in the state (1.139),

$$\Big\langle 0\Big|\hat{S}^{(1)}_a\Big|0\Big\rangle = \Big\langle 0\Big|\hat{S}^{(2)}_b\Big|0\Big\rangle = 0, \quad \Big\langle 0\Big|\hat{S}^{(1)}_a \hat{S}^{(2)}_b\Big|0\Big\rangle = -\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{4}. \tag{1.141}$$

We now find the probability $p_{a,b}(\epsilon^{(1)}_a; \epsilon^{(2)}_b)$ that the outcomes of measurements on the component of spin 1 in the direction $\boldsymbol{a}$ and that of 2 in the direction $\boldsymbol{b}$ are, respectively, the eigenvalues $\epsilon^{(1)}_a/2$ and $\epsilon^{(2)}_b/2$ ($\epsilon^{(1)}_a, \epsilon^{(2)}_b = \pm 1$). Note that those two measurements are compatible as the corresponding operators, being the operators acting on two different spins, commute. On using (1.60), that probability may be shown to be given by

$$p_{a,b}\left(\epsilon^{(1)}_a; \epsilon^{(2)}_b\right) = \left\langle 0\middle|\left(\frac{1}{2} + \epsilon^{(1)}_a \hat{S}^{(1)}_{\boldsymbol{a}}\right)\left(\frac{1}{2} + \epsilon^{(2)}_b \hat{S}^{(2)}_{\boldsymbol{b}}\right)\middle|0\right\rangle. \tag{1.142}$$

Apply (1.141) to get

$$p_{a,b}(\epsilon^{(1)}_a; \epsilon^{(2)}_b) = \frac{1}{4}\Big[1 - \epsilon^{(1)}_a \epsilon^{(2)}_b \boldsymbol{a}\cdot\boldsymbol{b}\Big]. \tag{1.143}$$

From this we infer that the probability of finding two spins in the same direction is zero, i.e. if $\boldsymbol{a} \cdot \boldsymbol{b} = 1$ then $p_{a,b}(+,+) = p_{a,b}(-,-) = 0$.

From the point of view of the discussion to follow, we consider the probabilities for the pairs of directions from a set of three directions $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$. Use (1.143), to shown that

$$p_{a,b}(+,+) + p_{b,c}(+,+) - p_{a,c}(+,+) = \frac{1}{2}\Big[\sin^2\left(\frac{\theta_{ab}}{2}\right) + \sin^2\left(\frac{\theta_{bc}}{2}\right) - \sin^2\left(\frac{\theta_{ac}}{2}\right)\Big], \tag{1.144}$$

where $\theta_{ab}, \theta_{bc}, \theta_{ac}$ are the angles between the directions identified by the respective subscripts.

With this we conclude the discussion of the methods of quantum mechanics relevant to us. Next we turn our attention to the important issue of identifying the characteristic non-classical features of the quantum theory.

1.7 Wave–Particle Duality

Classical mechanics deals with two types of dynamical systems, *particles* and *fields*. A particle is an entity localized in space and time. A field, on the other hand, is described by a function $E(x,t)$, called the *field amplitude* defined at a continuum of points in space. An important property of the fields is that the resultant amplitude due to two fields at a space-time point is the sum of their amplitudes:

$$E(x,t) = E_1(x,t) + E_2(x,t). \tag{1.145}$$

This is the *principle of superposition* of the field amplitudes. The field amplitude may be expressed as a Fourier series in time. Consider a field whose amplitude has the Fourier expansion

$$E(x,t) = \frac{1}{\sqrt{2}}\Big[A(x)\exp(-\mathrm{i}\omega t) + A^*(x)\exp(\mathrm{i}\omega t)\Big], \tag{1.146}$$

involving only one frequency. This describes a wave of frequency ω. A quantity of experimental interest is the *intensity* of the field. The intensity of the wave of a given frequency is the average of the modulus square of its amplitude over a period. The intensity of the wave described by (1.146) is thus $I(x,t) = |A(x)|^2$. Consider the superposition of two waves of the same frequency. On expressing each of the amplitudes $E_1(x,t)$ and $E_2(x,t)$ as in (1.146), the intensity of the superposed waves is seen to be given by

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\,\cos(\phi(x)), \tag{1.147}$$

where $I_i = |A_i(x)|^2$, $A_i(x) = |A_i(x)|\exp(\mathrm{i}\phi_i(x))$ is the intensity and amplitude of the individual waves, and $\phi(x) = \phi_1(x) - \phi_2(x)$ is the phase difference between them. This shows that the intensity I due to superposition of two waves is not a simple sum of the intensities I_1 and I_2 of the superposed waves; it is modified by the addition of an *interference* term which is the last term in

(1.147). There is no interference of this kind in the classical formalism if the entities involved are particles. The phenomenon of interference in classical mechanics is a characteristic of waves.

Now, recall that the quantum theory describes the state of motion of a particle in terms of a state vector and, like waves, the quantities of experimental interest in quantum mechanics, the expectation values, are quadratic in the state vector. Also, like the wave amplitudes, the state resulting from combination of states is a linear superposition of the corresponding state vectors. Hence, like waves, the expectation values of the observables in a state which is a linear superposition of two states may exhibit the phenomenon of interference. In other words, in quantum theory, a system can exhibit dual character: the particle-like character of space-time localization and the wave-like character of interference. The *wave-particle duality*, however, turns out to be *complementary*. It means that in an experiment a system would exhibit either the particle-like or the wave-like character but not both. We elaborate on these issues in the following.

For the sake of simplicity, consider a system described by the state vector $|\psi\rangle$ which is a superposition of the states $|\psi_i\rangle$ $(i = 1, 2)$ i.e.

$$|\psi\rangle = |\psi_1\rangle + |\psi_2\rangle. \tag{1.148}$$

The expectation value of an observable $\hat{A}$ in this state is given by

$$\langle\hat{A}\rangle = \langle\psi|\hat{A}|\psi\rangle = \langle\psi_1|\hat{A}|\psi_1\rangle + \langle\psi_2|\hat{A}|\psi_2\rangle + 2|\langle\psi_1|\hat{A}|\psi_2\rangle|\cos(\phi), \tag{1.149}$$

where ϕ is the phase of the complex number $\langle\psi_1|\hat{A}|\psi_2\rangle$. This shows that the expectation value of an observable $\hat{A}$ in a state $|\psi\rangle$, which is formed by a linear superposition of the states $|\psi_1\rangle$ and $|\psi_2\rangle$ is not a simple sum of its expectation values in the states $|\psi_1\rangle$ and $|\psi_2\rangle$ alone but is modified by the addition of an interference term (which is the last term in (1.149)). Clearly, the interference effects may be exhibited if the observable has non-zero matrix element between the two states. The observable may, for example, be $|a\rangle\langle a|$ where $|a\rangle$ is an eigenstate of $\hat{A}$ with eigenvalue a. The expectation values in (1.149) are then the probabilities of observing the system in state $|a\rangle$.

The probability amplitudes in quantum mechanics thus exhibit the interference phenomenon of waves. In order to understand the origin of quantum interference, note that the fact that the state $|\psi\rangle$ is a superposition of two states means that the system can exist in one or the other state. We will see that interference is due to the lack of information about the state in which the system existed at the time of observation. In order to see that, let the system, before it is observed, pass through a detector whose state is changed differently on interaction with the two superposed states. Let $|d_0\rangle$ be the state of the detector before interaction and let $|d_i\rangle$ $(i = 1, 2)$ be its state after interaction with the state labeled i. The combined state of the observed system (hereafter called system S) and the detector after the interaction may be expressed as

$$|\Psi\rangle = |\psi_1\rangle|d_1\rangle + |\psi_2\rangle|d_2\rangle, \tag{1.150}$$

We are, however, interested in the results of measurements on the system S alone. As explained in Sect. 1.5, those results are determined by the density operator $\hat{\rho}_s = \mathrm{Tr_d}[|\Psi\rangle\langle\Psi|]$ where $\mathrm{Tr_d}$ denotes trace over the detector states. Verify that

$$\hat{\rho}_\mathrm{s} = |\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2| + |\psi_1\rangle\langle\psi_2|\langle d_2|d_1\rangle + |\psi_2\rangle\langle\psi_1|\langle d_1|d_2\rangle, \tag{1.151}$$

The expectation value of a system observable $\hat{A}$ alone is then given by

$$\langle\hat{A}\rangle = \langle\psi_1|\hat{A}|\psi_1\rangle + \langle\psi_2|\hat{A}|\psi_2\rangle + 2\mathrm{Re}\Big(\langle\psi_1|\hat{A}|\psi_2\rangle\langle d_1|d_2\rangle\Big). \tag{1.152}$$

The interference term in this case has an additional factor $\langle d_1|d_2\rangle$. This factor determines the overlap between the states in which the detector is left as a result of its interaction with the two superposed states of the system. Since $|\langle d_1|d_2\rangle| \leq 1$, the process of detection reduces the contribution of the interference term. The two superposed states are evidently distinguishable unambiguously if the states in which they leave the detector on interaction with it are orthogonal, i.e. if $\langle d_1|d_2\rangle = 0$. The interference term in (1.152) then vanishes. This shows that there is no interference if the superposed state in which the system existed before the time of observation is known. In other words, the interference is lost if we know by *which path* (*welcher weg* in German) the system arrived at the detector. Now, in the classical mechanical description, a particle follows a definite trajectory or path. Hence, knowing unambiguously the path of an object is a particle-like property. From this it may, therefore, be inferred that the interference, which is a wave-like phenomenon, is lost if the path, which is a particle-like characteristic, is known. Hence, though what is perceived as a particle in classical mechanics, may exhibit wave-like interference phenomenon, the wave-particle duality is complementary. We will see in Chap. 5 that even what are perceived as waves in classical mechanics may exhibit particle-like dual character.

The loss of interference due to the process of detection may at times be attributed to a change in the observed part of the superposed states brought about by their interaction with the detector. The details of the changes so brought about depend upon the particular situation at hand. However, it is at times possible to conceive detectors which reveal which-path information by interacting with that operator of the system which commutes with the one under observation. Such detectors, therefore, do not alter the observed part of the state of the system. The loss of interference in that case can not be attributed to any detector influenced state alteration mechanism. An example of such a detector in the context of quantum optics may be found in [13].

In the following, we refer to the states $|d_i\rangle$ of the detector resulting from its interaction with the superposed states of the system under observation as the *which-path states.*

We have seen above how the process of detection may reduce the effect of interference exhibited by observations made on a system in a superposed

state. Let us now examine the possibility of observing interference in a joint measurement of a system observable $\hat{A}$ and a detector observable $\hat{D}$. In the combined state of the system and the detector, given by (1.150), the expectation value in question is evidently

$$\langle\Psi|\hat{D}\hat{A}|\Psi\rangle = \sum_{i=1}^{2} |\langle\psi_i|\hat{A}|\psi_i\rangle|^2 |\langle d_i|\hat{D}|d_i\rangle|^2 + 2\mathrm{Re}\Big[\langle\psi_1|\hat{A}|\psi_2\rangle\langle d_1|\hat{D}|d_2\rangle\Big]. \tag{1.153}$$

This implies that the joint observation of the system observable $\hat{A}$ and the detector observable $\hat{D}$ would exhibit interference effects between the superposed states $|\psi_1\rangle$ and $|\psi_2\rangle$ if $\langle\psi_1|\hat{A}|\psi_2\rangle \neq 0$ i.e. if $\hat{A}$ exhibits interference in the absence of detection, and if $\langle d_1|\hat{D}|d_2\rangle \neq 0$ i.e. if $\hat{D}$ has a non-vanishing matrix element between which-path detector states. Recall from (1.152) that if the which-path detector states are orthogonal to each other then the system observable $\hat{A}$ alone would not exhibit interference effects. However, its joint observation with a detector observable $\hat{D}$ may exhibit such an effect even if $\langle d_1|d_2\rangle = 0$ but if $\langle d_1|\hat{D}|d_2\rangle \neq 0$. Since the condition $\langle d_1|d_2\rangle = 0$ provides which-path information, the condition $\langle d_1|\hat{D}|d_2\rangle \neq 0$ means that the detector observable $\hat{D}$ is an eraser, called the *quantum eraser*, of the which-path information [14]. It should be emphasized that the interference in this case is not in $\hat{A}$ but in joint observation of $\hat{A}$ with the detector observable $\hat{D}$. For examples of the possibility of realization of quantum eraser in the context of quantum optics, see [14, 15].

1.8 Measurement Postulate and Paradoxes of Quantum Theory

The validity of a postulate of a physical theory is judged, of course, by its ability to describe the observed physical phenomena. However, in spite of its success on this count, the postulate (postulate 4) which relates the theoretical predictions with experimental observations, has been at the centre of controversy right since its conception. The controversy revolves mainly around the questions involving (a) that postulate's consistency with other postulates, and (b) its denial of the objective reality. Those issues are brought out forcefully in the thought experiment (*Gedankenexperiment* in German) of Schrödinger and that of Einstein, Podolsky and Rosen (EPR). In the following we outline briefly first the issue of consistency of the process of measurement envisaged by the postulate in question with other postulates of the quantum theory and follow it up with a discussion of the two paradoxes: the ***Schrödinger's Cat Paradox*** and the ***EPR Paradox***.

1.8.1 The Measurement Problem

A measurement is an outcome of interaction between the system under observation (hereafter called the *object*) and an apparatus designed to measure an observable. Since there is no restriction in the postulates of the quantum theory on the kind of physical systems that it seeks to describe, it is expected to describe an apparatus as well as it describes the object. If that be so, the process of measurement envisaged by the quantum theory should be consistent with its other postulates applied to the interacting system of the object and the apparatus. In order to examine whether or not it is so, consider the act of measurement of an observable $\hat{B}$. Let $\{|b_i\rangle\}$ be the set of orthonormal eigenvectors of $\hat{B}$ corresponding to the eigenvalues $\{b_i\}$ $(i = 1, \ldots, n)$ where, for the sake of simplicity, we assume that those eigenvectors are denumerable and the eigenvalues non degenerate (see Chap. 10). Any state of the object is then expressible as

$$|\psi\rangle = \sum_{i=1}^{n} \alpha_i |b_i\rangle. \tag{1.154}$$

Similarly, let the state space of the apparatus be spanned by the orthonormal set $\{|a_i\rangle\}$ and let the apparatus be initially in a pure state. Recall that, according to the measurement postulate, the state of the object on interaction with the apparatus is reduced to one of the eigenstates, say $|b_k\rangle$, of the observable being measured with probability $|\langle\psi|b_k\rangle|^2 = |\alpha_k|^2$ and that the outcome of the measurement is the corresponding eigenvalue b_k. This implies that the state of the combined system of the object and the apparatus after the interaction be described by the density operator

$$\hat{\rho} = \sum_{i=1} |\alpha_i|^2 |a_i, b_i\rangle\langle a_i, b_i|. \tag{1.155}$$

For details see the papers in [11]. The density operator (1.155) characterizes a mixture of states $\{|a_i, b_i\rangle\}$ with probability $\{|\alpha_i|^2\}$. However, recall from Sect. 1.5 that a pure state remains pure under a unitary transformation. Since the initial state of the combined system is a pure state whose evolution is governed by unitary Schrödinger evolution operator, the state of the combined system at any time will be pure. Note also that the state reduction is an irreversible process whereas a unitary evolution is reversible. The measurement postulate thus seems to be out of tune with other postulates. The non-unitary, irreversible, evolution may, however, arise in the limit of non-denumerably infinite number of degrees of freedom of the environment with which a system interacts (see Chap. 8).

Quantum theory thus seems to have different laws depending upon whether a particular process is a measurement or not. What, then, distinguishes an apparatus from any other quantum mechanical system? Although the quantum theory does not provide an answer to it, it is generally accepted that an apparatus is a *macroscopic* system. Each macroscopically dis-

tinguishable state is a statistical mixture of several microscopic states. Hence a macroscopic state should be described, not by a state vector, but by a density operator. However, even if it is assumed that the apparatus is initially in a mixed state, the Schrödinger evolution still does not lead to the form (1.155) expected of a process of measurement [11].

The cause of the measurement paradox thus seems to be rooted in linearity (resulting from superposition principle) and the unitarity of the quantum evolution. It would thus appear that by postulating a different kind of evolution when a quantum system interacts with an apparatus may resolve the paradox. Indeed, a non-linear time-evolution for object-apparatus has been advocated by some authors [11]. However, a theory whose laws depend upon whether or not a system is an apparatus can not be regarded as satisfactory.

Finally, it may be argued that the state reduction is not a physical process, that the quantum theory predicts only the statistical average of many observations and that state reduction postulate is needed to make statistical predictions about the outcome of individual experiments. The state vector provides just a means of calculating the statistical outcomes of the process of measurement made on identically prepared systems. It describes an ensemble of identically prepared systems and not any individual system. The paradoxes discussed below indeed bring out emphatically the inadequacy of the state vector in describing the outcome of individual measurements.

1.8.2 Schrödinger's Cat Paradox

Following Schrödinger [16], assume that a cat is pinned up in a steel chamber. Let there be a mechanism (details of which are unimportant for the present) which releases a poisonous gas in the chamber when hit by a spin pointing in the +z-direction but not if the spin points in the -z-direction. The release of the poisonous gas is assumed to kill the cat. Let the said mechanism be hit by a spin-1/2 whose states in the $\pm$z-directions are denoted by $|\pm\rangle$. Let $|d\rangle$ denote the state of the dead cat and $|l\rangle$ that of the living cat. If a spin in state $|+\rangle$ hits the triggering mechanism then quantum mechanics predicts that the state of the combined system of the cat and the spin is $|+, d\rangle$. In any subsequent experiment, the cat will be found dead with certainty. Similarly, if that mechanism is hit by a spin in state $|-\rangle$ then the quantum mechanical state of the combined system of the cat and the spin would be $|-, l\rangle$, and in any subsequent experiment the cat will be found alive with certainty. Those predictions agree with what is the common perception of reality or the objective reality.

The situation is, however, different if the triggering mechanism is hit with a spin in a superposition state, say, in the state $\sqrt{1/2}(|+\rangle - |-\rangle)$. According to quantum mechanics, the state of the combined system of the cat and the spin would be

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\Big(|+, d\rangle - |-, l\rangle\Big). \tag{1.156}$$

This state is a superposition of the states in one of which the cat is dead and in the other in which the cat is alive with equal probability. According to the measurement postulate, the cat in a process of measurement will be found dead or alive with equal probability and that no statement can be made about the state of the cat until a measurement is performed, i.e. the state of the cat is *decided* by the act of measurement. The act of measurement could be the act of looking at the cat after the event is over. Till then, the cat remains in a state of suspended animation! This is contrary to the objective reality according to which the state of the cat is decided according to how the spin hits the triggering mechanism and not by any subsequent act of measurement. The subsequent act of measurement only *ascertains* that state. It would, therefore, seem that a macroscopic object, like a cat, may not exist in a superposition state. A state that is a superposition of macroscopic states is often referred to as a *cat state.* Which quantum states should we accept as macroscopic though may be a matter of debate. However, the coherent states (introduced in Chap. 3) of a harmonic oscillator and those of spins are widely recognized to be macroscopic. It is because those states are most classical; in fact the only pure states of respective systems possessing classical properties (in the sense explained in Chap. 4). A number of models for generating coherent superposition of field coherent states [17] and those of spin states [18] in the context of quantum optics have been proposed.

1.8.3 EPR Paradox

Here we present experimentally important Bohm's [19] version of the EPR paradox [20]. It considers a maximally entangled state (1.139) of two spin-1/2s. Now, let the two spins fly apart and let the z-component of spin 1 be determined experimentally after it ceases to interact with the spin 2. According to the measurement postulate, spin 1 then collapses to an eigenstate of its z-component. Since the combined state of the spins is given by (1.139), it follows that if the state of spin 1 is found to be $|+\rangle$ ($|-\rangle$) then that of spin 2 is certainly $|-\rangle$ ($|+\rangle$). According to the measurement postulate, the state of spin 2 in this case is reduced to an eigenstate of its z-component.

However, we can, instead, measure some other component, say, the x-component of 1. The eigenstates $|\pm, x\rangle$ of the x-component of a spin-1/2 are related with those of its z-component by the relation

$$|\pm, x\rangle = \sqrt{\frac{1}{2}}\Big[|+\rangle \pm |-\rangle\Big]. \tag{1.157}$$

The reader may verify by expressing $\hat{S}_x$ in terms of $\hat{S}_\pm$ that these are indeed the eigenstates of $\hat{S}_x$. Invert this relation to express $|\pm\rangle$ in terms of $|\pm, x\rangle$. Substitute the resulting expression in (1.139) to show that the state of the spins assumes the form

$$|0\rangle = \sqrt{\frac{1}{2}}\Big[|-, +, x\rangle - |+, -, x\rangle\Big]. \tag{1.158}$$

Hence, if the outcome of a measurement of the x-component of spin 1 is $1/2$ $(-1/2)$ then it follows that the spin 2 is certainly in the state $|-,x\rangle$ $(|+,x\rangle)$. According to the measurement postulate, the state of spin 2 in this case is reduced to an eigenstate of its x-component.

Observe that the state reduction of spin 2 in the example above is caused, not by a measurement performed on it, but by the one performed on another spin not interacting with it at the time of the measurement. This shows that the process of state reduction of a spin may be controlled by the process of measurement performed on another spin with which it interacted in the past but with which it has no interaction at the time of measurement. Since none of the spins knows while it interacts with the other, which of its component is going to be measured by a subsequent act of measurement, the said controlling influence by the process of measurement envisaged by the quantum theory is *non-local.*

Moreover, an experiment can be performed to determine simultaneously one component of spin 1 and the same or the other component of the spin 2. This is permissible because all the operators corresponding to one spin commute with all the operators corresponding to the other and, according to the quantum theory, the value of the observables whose corresponding operators commute can be determined simultaneously with certainty. Let then the z-component of spin 1 and the x-component of spin 2 be measured simultaneously. As discussed above, the knowledge of the z-component of spin 1 provides precise information about the state of that component of spin 2 as well. However, the x-component of spin 2 is also known precisely by virtue of the measurement performed on it. We thus know simultaneously the precise values of two non-commuting observables. That is in contradiction with the quantum theoretical dictum that two non-commuting observables can not possess precise values simultaneously.

This thought experiment exemplifies a situations wherein, it is possible, in principle, to determine with certainty the values of two non-commuting operators simultaneously. As mentioned above, this is incompatible with the description of a system in terms of a state vector. Therefore, EPR [20] argue that, the description in terms of a state vector is incomplete. According to them, a necessary requirement for the description given by a theory to be *complete* is that *every element of the physical reality must have a counterpart in the physical theory.* Their criterion for determining the elements of physical reality is that *if, without in any way disturbing a system, we can predict with certainty (i.e. with probability unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.* Since, in the example above, it is possible to determine with certainty the values of two non-commuting components of a spin without disturbing it, there is an element of physical reality corresponding to those spin components. However, according to the description of the spin in terms of a state vector, the values of non-commuting operators can not be determined

simultaneously with certainty. Hence, the said elements of the physical reality have no counterpart in the description in terms of a state vector. Such a description is, therefore, incomplete. They left open the question of whether or not a complete description exists but concluded by saying that they believe that it does.

An attempt in the direction of providing a complete description of a physical system is in terms of the hidden variables theory discussed next.

1.9 Local Hidden Variables Theory

The paradoxes discussed above may be resolved by assuming that in each realization the system carries definitive value of all its observables distributed among different realizations with the probability assigned by the measurement postulate. This is the basic premise of the hidden variables theory. It attributes the quantum indeterminism to some 'hidden' influence analogous to that due to microscopic constituents in a macroscopic classical statistical system. It assumes that, with a complete knowledge of those hidden variables, the outcome of the observations can be predicted with certainty and that the probabilistic nature of the quantum predictions is a result of our ignorance of such variables. If it is also assumed that the measuring devices at two different places are neither correlated with each other nor with the object under observation then the hidden variables theory is called a *local hidden variables theory* (LHVT). We do not discuss the hidden variables theory in all its details (for a survey, see [21]) and restrict ourselves to highlighting the implications of the LHVT only.

The incompatibility of the LHVT with the quantum theory can be seen at the outset. For, it implies that in each realization of an experiment the system carries definitive information about every observable whether commuting or not. According to quantum mechanics, a definitive information about non-commuting observables is precluded. Thus, in the hidden variables theory, we can define joint probability distributions involving even non-commuting observables. In the quantum theory, there is no concept of a joint probability distribution involving non-commuting observables. Let us, nevertheless, investigate some observable consequences of the LHVT and compare them with the predictions of the quantum theory.

The simplest and the most widely studied system, as regards the verifiable consequences of the LHVT are concerned, is a system of two spin-1/2 particles in a maximally entangled state such as (1.139). For the two spins in that state, we have in (1.144) the quantum theoretic relationship between $p_{a,b}(+,+)$, $p_{b,c}(+,+)$ and $p_{a,c}(+,+)$ where $p_{a,b}(\epsilon_a, \epsilon_b)$ is the probability that the outcome of measurement on the component of spin 1 in the direction $\boldsymbol{a}$ and that of spin 2 in the direction $\boldsymbol{b}$ are, respectively, the eigenvalues $\epsilon_a/2$ and $\epsilon_b/2$ ($\epsilon_a, \epsilon_b = \pm 1$). Here we derive that relationship following the local hidden variables theory.

To that end, let $P\left(\epsilon_a^{(1)}, \epsilon_b^{(1)}, \epsilon_c^{(1)}; \epsilon_a^{(2)}, \epsilon_b^{(2)}, \epsilon_c^{(2)}\right)$ be the joint probability that the spin 1 has the components measuring $\epsilon_a^{(1)}/2, \epsilon_b^{(1)}/2, \epsilon_c^{(1)}/2$ along the directions $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and that $\epsilon_a^{(2)}/2, \epsilon_b^{(2)}/2, \epsilon_c^{(2)}/2$ are the components of the spin 2 along the same directions where each of the ϵ assumes values ± 1. Now, the probability of finding two spins in the same direction, say the direction $\boldsymbol{a}$, may be expressed in terms of the joint probability introduced above as

$$p_{a,a}\left(\epsilon_a; \epsilon_a\right) = \sum P\left(\epsilon_a, \epsilon_b^{(1)}, \epsilon_c^{(1)}; \epsilon_a, \epsilon_b^{(2)}, \epsilon_c^{(2)}\right), \tag{1.159}$$

where the summation is over all the $\epsilon^{(1)}$'s and the $\epsilon^{(2)}$'s. If the spins are in the state (1.139) then, by virtue of (1.143), $p_{a,a}(\epsilon_a; \epsilon_a) = 0$. Since classical probabilities are non-negative, it follows that $p_{aa}(\epsilon_a; \epsilon_a) = 0$ can hold only if each of the joint probability under the summation in (1.159) is zero. On using the same argument for the probabilities in the other two directions, it follows that the $P's$ with the same sign at the same place on the two sides of the semicolon is zero i.e.

$$\begin{aligned} P\left(\epsilon, \epsilon_b^{(1)}, \epsilon_c^{(1)}; \epsilon, \epsilon_b^{(2)}, \epsilon_c^{(2)}\right) &= P\left(\epsilon_a^{(1)}, \epsilon, \epsilon_c^{(1)}; \epsilon_a^{(2)}, \epsilon, \epsilon_c^{(2)}\right) \\ &= P\left(\epsilon_a^{(1)}, \epsilon_b^{(1)}, \epsilon; \epsilon_a^{(2)}, \epsilon_b^{(2)}, \epsilon\right) = 0. \end{aligned} \tag{1.160}$$

Now, note that $p_{a,b}(\epsilon_a^{(1)}; \epsilon_b^{(2)})$ is obtained from $P\left(\epsilon_a^{(1)}, \epsilon_b^{(1)}, \epsilon_c^{(1)}; \epsilon_a^{(2)}, \epsilon_b^{(2)}, \epsilon_c^{(2)}\right)$ by summing over all the $\epsilon' s$ other than $\epsilon_a^{(1)}$ and $\epsilon_b^{(2)}$. Similar argument for $p_{b,c}\left(\epsilon_b^{(1)}; \epsilon_c^{(2)}\right)$ and $p_{a,c}\left(\epsilon_a^{(1)}; \epsilon_c^{(2)}\right)$ along with the use of (1.160) yields

$$\begin{aligned} p_{a,b}(+;+) &= P(+,-,+;-,+,-) + P(+,-,-;-,+,+), \\ p_{b,c}(+;+) &= P(+,+,-;-,-,+) + P(-,+,-;+,-,+), \\ p_{a,c}(+;+) &= P(+,+,-;-,-,+) + P(+,-,-;-,+,+). \end{aligned} \tag{1.161}$$

These lead to the relation

$$\begin{aligned} &p_{a,b}(+;+) + p_{b,c}(+;+) - p_{a,c}(+;+) \\ &= P(+,-,+;-,+,-) + P(-,+,-;+,-,+). \end{aligned} \tag{1.162}$$

Again, positivity of the probabilities on the right hand side of (1.162) implies the inequality

$$p_{a,b}(+;+) + p_{b,c}(+;+) - p_{a,c}(+;+) \geq 0 \tag{1.163}$$

known as *Bell's inequality* [22]. Its derivation given here is due to Wigner [23]. Recall that the corresponding quantum theoretic result is the equality (1.144). That equality need not respect the inequality (1.163). A violation of (1.163) for any choice of directions would constitute a rebuttal of LHVT.

It is not difficult to see that if, for example, the three vectors are coplanar and $\theta_{ab} = \theta_{bc} = \pi/3$, $\theta_{ac} = 2\pi/3$, then the value of (1.144) is $-1/4$ which violates (1.163).

It has been shown that any pure entangled state of two spin-1/2s violates corresponding Bell's inequality [12, 24, 25]. Similar inequalities involving four directions have also been derived [26]. A more powerful criterion for comparing the predictions of the two theories in the form of an equality is obtained for a system of three spin-1/2s [27, 28].

Violations of Bell-type inequalities have been observed experimentally (see the papers reprinted in [11]). They thus establish that the LHVT can not account for quantum correlations. In other words, the nature of quantum indeterminism is not the same as that of a classical statistical mechanical system.

Note that the assumptions involved in deriving the results of hidden variables theory above are locality and the positivity of the probabilities. By lifting any one of those restrictions we can make the hidden variables theory agree with the quantum theoretic predictions. However, relaxing the requirement of locality would mean that the states of the measuring devices are correlated with the states of the object and with each other. It has the questionable feature of denying a description of isolated systems. The *negative probabilities* [29] are, of course, unphysical. However, as discussed in Chap. 4, they can be useful as calculational tools and in identifying signatures of quantum effects.

2. Algebra of the Exponential Operator

The exponential operator, i.e. the exponential function of an operator, defined in (1.19) by way of a series expansion, is of paramount interest in mathematical physics. In view of its importance, we discuss in this chapter some useful algebraic operations involving an exponential operator. The operations discussed here are (i) the parametric differentiation of an exponential operator, (ii) its reduction to a polynomial when the exponent is finite-dimensional, (ii) similarity transformation, defined below, by an exponential operator, (iii) the operation of disentangling an exponential, and (iv) evaluation of time-ordered exponential integral.

2.1 Parametric Differentiation of the Exponential

Consider the exponential function $\exp(\hat{A}(t))$ of an operator which is a function of a scalar t. In this section we evaluate its derivative with respect to t. To that end, use the series expansion (1.19) of the exponential and the formula (1.40) for the derivative of $\hat{A}^m(t)$ with respect to t to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\exp[\hat{A}(t)] = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{(n+1)!}\hat{A}^k(t)\frac{\mathrm{d}\hat{A}(t)}{\mathrm{d}t}\hat{A}^{n-k}(t). \tag{2.1}$$

Now, (i) interchange the n and k summations in (2.1) using

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} f_{n,k} = \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} f_{n,k} = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} f_{n+k,k}, \tag{2.2}$$

(ii) multiply and divide the resulting summand by $n!k!$, (iii) rewrite $n!k!/(n+k+1)!$ in terms of an integral using (A.18), and finally (iv) carry the summations over n and k to arrive at the *Sneddon's formula*

$$\frac{\mathrm{d}}{\mathrm{d}t}\exp[\hat{A}(t)] = \int_0^1 \exp[u\hat{A}]\frac{\mathrm{d}\hat{A}}{\mathrm{d}t}\exp[(1-u)\hat{A}]\mathrm{d}u. \tag{2.3}$$

If $\hat{A}(t) = \hat{A}t$ then this yields the familiar c-number result

$$\frac{\mathrm{d}}{\mathrm{d}t}\exp(\hat{A}t) = \hat{A}\exp(\hat{A}t) = \exp(\hat{A}t)\hat{A}. \tag{2.4}$$

2.2 Exponential of a Finite-Dimensional Operator

In this section we show that the exponential of an n-dimensional operator $\hat{X}$ may be expressed as a polynomial of degree at most $n-1$ in $\hat{X}$. To that end, we recall eq.(10.24) which states that an n-dimensional operator satisfies the equation

$$\prod_{i=1}^{n}\left(\hat{X}-\lambda_i\right)=0, \tag{2.5}$$

λ_i being the eigenvalues of $\hat{X}$. This is a polynomial of degree n in $\hat{X}$. Recall also that the equation (2.5) is not necessarily the minimum polynomial equation of $\hat{X}$. Let $N \leq n$ be the degree of the minimum polynomial equation of $\hat{X}$. We may rewrite such an equation in the form

$$\hat{X}^N=\sum_{m=0}^{N-1}\alpha_m\hat{X}^m. \tag{2.6}$$

Here α_m are known in terms of the λ's. Using (2.6), we can express the n^{th} and higher powers of $\hat{X}$ as linear combinations of its powers up to $N-1$. Consequently, we may write

$$\exp(\theta\hat{X})=\sum_{m=0}^{N-1}C_m(\theta)\hat{X}^m. \tag{2.7}$$

Note that

$$C_k(0)=\delta_{k0}. \tag{2.8}$$

To detrmine the unknown functions $C_m(\theta)$, differentiate (2.7) with respect to θ to obtain

$$\sum_{m=1}^{N}C_{m-1}(\theta)\hat{X}^m=\sum_{m=0}^{N-1}\dot{C}_m(\theta)\hat{X}^m. \tag{2.9}$$

The dot over a quantity denotes derivative with respect to θ. Now, use (2.6) for $\hat{X}^N$ and compare equal powers of $\hat{X}$ in (2.9) to arrive at the equation

$$\dot{C}_k(\theta)=\alpha_kC_{N-1}(\theta)+C_{k-1}(\theta),\quad C_{-1}(\theta)=0, \tag{2.10}$$

The set of equations generated by (2.10) for $k=0,\ldots,N-1$ determine C_k. Their solution, with (2.8) as the initial condition, on substitution in (2.7) gives the desired expression of the exponential as a polynomial.

As an example, consider a spin-1/2. Any of its component $\hat{S}_a$ along a direction $\boldsymbol{a}$ is a two-dimensional operator ($n=2$) which can assume the values $\pm 1/2$. The equation (2.5) in this case reads

$$\hat{S}_a^2=\frac{1}{4}. \tag{2.11}$$

It is also the minimum polynomial equation for $\hat{S}_a$. On comparing this with (2.6) we note that $\alpha_0 = 1/4$ and $\alpha_1 = 0$. Since $N = n = 2$, the equations (2.10) reduce to

$$\dot{C}_0(\theta) = \frac{1}{4}C_1(\theta), \qquad \dot{C}_1(\theta) = C_0(\theta). \tag{2.12}$$

The solution of these, with the condition (2.8), on substitution in (2.7) yields

$$\exp(\theta\hat{S}_a) = \cosh\left(\frac{\theta}{2}\right) + 2\sinh\left(\frac{\theta}{2}\right)\hat{S}_a. \tag{2.13}$$

We leave it as an excersise for the reader to show that if $\hat{J}_a$ is a component of a spin-1 then

$$\exp(\theta\hat{J}_a) = 1 + \sinh(\theta)\hat{J}_a + (\cosh(\theta) - 1)\hat{J}_a^2. \tag{2.14}$$

This may be established by noting that the eigenvalues of $\hat{J}_a$ are $0, \pm 1$ and that the equation (2.5) in this case assumes the form $\hat{J}_a(\hat{J}_a^2 - 1) = 0$.

2.3 Lie Algebraic Similarity Transformations

Let $\hat{S}$ be a non-singular operator. The transformation defined by

$$\hat{S}^{-1}\hat{A}\hat{S} = \hat{B} \tag{2.15}$$

is called the *similarity transformation* of $\hat{A}$ by $\hat{S}$. By invoking the definition of the power of an operator we see that, for a positive integer m,

$$\left[\hat{S}^{-1}\hat{A}\hat{S}\right]^m = \hat{S}^{-1}\hat{A}^m\hat{S}. \tag{2.16}$$

As a consequence of this we note that if $F(\hat{A})$ is a function expandable in a power series of $\hat{A}$ then

$$\hat{S}^{-1}F\left(\hat{A}\right)\hat{S} = F\left(\hat{S}^{-1}\hat{A}\hat{S}\right). \tag{2.17}$$

As an application of the similarity transform, note that if $\hat{S}^{-1}$ exists then we can shuffle $\hat{S}$ to the right (left) in the product $\hat{S}\hat{A}$ $(\hat{A}\hat{S})$ by means of the relations

$$\hat{S}\hat{A} = (\hat{S}\hat{A}\hat{S}^{-1})\hat{S}, \qquad \hat{A}\hat{S} = \hat{S}(\hat{S}^{-1}\hat{A}\hat{S}) \tag{2.18}$$

involving a similarity transformation.

Of particular interest to us is the similarity transformation (2.15) when $\hat{S} = \exp(\hat{P})$. As an important example, recall from (1.77) that if a system is described by a time-independent hamiltonian $\hat{H}$ then the time-evolution of an operator $\hat{A}$ is determined by the similarity transformation $\hat{A}(t) = \exp(\mathrm{i}\hat{H}t/\hbar)\hat{A}\exp(-\mathrm{i}\hat{H}t/\hbar)$. The similarity transformation by an exponential operator may be expressed as

$$\hat{A}(\theta) \equiv \exp\left(-\theta\hat{P}\right)\hat{A}\exp\left(\theta\hat{P}\right)$$
$$= \hat{A} - \theta\left[\hat{P},\ \hat{A}\right] + \frac{\theta^2}{2!}\left[\hat{P},\ \left[\hat{P},\hat{A}\right]\right] + \cdots$$
$$\equiv \sum_{n=0}^{\infty}\frac{(-\theta)^n}{n!}\hat{\hat{L}}_P^{\,n}\hat{A} \ = \ \exp\left[-\theta\hat{\hat{L}}_P\right]\hat{A}, \tag{2.19}$$

$$\hat{\hat{L}}_P^{\,0}\hat{A} = \hat{A},\ \hat{\hat{L}}_P\hat{A} = \left[\hat{P},\ \hat{A}\right], \cdots, \hat{\hat{L}}_P^{\,n}\hat{A} = \hat{\hat{L}}_P^{\,n-1}[\hat{P},\hat{A}]. \tag{2.20}$$

To prove (2.19), show that $\hat{A}(\theta)$ obeys the differential equation $(d/d\theta)\hat{A}(\theta) = -\hat{\hat{L}}_P\hat{A}(\theta)$ whose solution is evidently (2.19).

The evaluation of (2.19) is generally a formidable task unless the series terminates at a low order. A systematic approach for its evaluation may be developed when $\hat{A}$ and $\hat{P}$ are the elements of a *Lie algebra*: A linear vector space of operators is said to constitute a Lie algebra if it is closed under the operation of commutation, i.e. if the commutator of any two operators in the space also lies in that space. Consider a vector space spanned by a complete set of operators $\hat{X}_1, \cdots, \hat{X}_n$. This space would constitute a Lie algebra if

$$\left[\hat{X}_i,\ \hat{X}_j\right] = \sum_{k=1}^{n} c_{ijk}\hat{X}_k, \tag{2.21}$$

c_{ijk} being c-numbers called the *structure constants* of the algebra. An algebra is characterized by its structure constants. The operators $\hat{X}_1, \cdots, \hat{X}_n$ are called the *generators* of the algebra.

Now, consider the similarity transformation

$$\hat{X}_i(\theta) = \exp\left(-\theta\hat{Z}\right)\hat{X}_i\exp\left(\theta\hat{Z}\right). \tag{2.22}$$

Differentiation of this equation with respect to θ yields

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\hat{X}_i(\theta) = \exp\left(-\theta\hat{Z}\right)\left[\hat{X}_i,\ \hat{Z}\right]\exp\left(\theta\hat{Z}\right). \tag{2.23}$$

Let $\hat{Z}$ be an element of the given algebra expressible as

$$\hat{Z} = \sum_{i=1}^{n}\alpha_i\hat{X}_i, \tag{2.24}$$

α_i being c-numbers. Substitution of (2.24) in (2.23) and use of the commutation relation (2.21) leads to a set of first order coupled ordinary linear differential equations for $\left\{\hat{X}_i(\theta)\right\}$:

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\hat{X}_i(\theta) = \sum_{k=1}^{n}\left[\sum_{j=1}^{n}c_{ijk}\alpha_j\right]\hat{X}_k(\theta). \tag{2.25}$$

Solution of (2.25) gives $\hat{X}_i(\theta)$ in terms of a linear combination of $\hat{X}_1(0), \cdots, \hat{X}_n(0) \equiv \hat{X}_1, \cdots, \hat{X}_n$.

In the next section we solve (2.25) explicitly for some algebras which encompass a number of hamiltonians of frequent occurrence in quantum optics.

2.3.1 Harmonic Oscillator Algebra

The set of operators $\left(\hat{a}^\dagger, \hat{a}, \hat{N} \equiv \hat{a}^\dagger\hat{a}, I\right)$ obeying the commutation relations

$$\left[\hat{a},\ \hat{a}^\dagger\right] = 1, \qquad \left[\hat{N},\ \hat{a}\right] = -\hat{a}, \qquad \left[\hat{N},\ \hat{a}^\dagger\right] = \hat{a}^\dagger \tag{2.26}$$

generates the *harmonic oscillator algebra.* It is so called because the hamiltonian of a harmonic oscillator (h.o.) coupled linearly to a driving force is expressible as a linear combination of the elements of this algebra. The last two commutation relations in (2.26) are consequences of the first relation called the *canonical or bosonic commutation relation.* The operators $\hat{a}$, $\hat{a}^\dagger$ are also called *boson operators.* An element $\hat{Z}_h$ in this algebra is expressible as

$$\hat{Z}_{\text{ho}} = \alpha_1\hat{a} + \alpha_2\hat{a}^\dagger + \alpha_3\hat{a}^\dagger\hat{a}, \tag{2.27}$$

the $\alpha's$ being c-numbers. Consider the similarity transformation

$$\hat{a}(\theta) = \exp\left(-\theta\hat{Z}_{\text{ho}}\right)\hat{a}\exp\left(\theta\hat{Z}_{\text{ho}}\right). \tag{2.28}$$

It is straightforward to show that

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\hat{a}(\theta) = \alpha_3\hat{a}(\theta) + \alpha_2. \tag{2.29}$$

This is solved by

$$\hat{a}(\theta) = \exp(\alpha_3\theta)\hat{a} + \frac{\alpha_2}{\alpha_3}\Big[\exp(\alpha_3\theta) - 1\Big]. \tag{2.30}$$

Next we derive results for some frequently encountered special cases of (2.30).

1. Set $\alpha_3 = 0,\ \theta = 1$ to reduce (2.30) to

$$\exp\left\{-\left(\alpha_1\hat{a} + \alpha_2\hat{a}^\dagger\right)\right\}\hat{a}\exp\left\{\left(\alpha_1\hat{a} + \alpha_2\hat{a}^\dagger\right)\right\} = \hat{a} + \alpha_2. \tag{2.31}$$

Rearrange this to arrive at the commutation relation

$$\left[\hat{a},\ \exp\left(\alpha_1\hat{a} + \alpha_2\hat{a}^\dagger\right)\right] = \alpha_2\exp\left(\alpha_1\hat{a} + \alpha_2\hat{a}^\dagger\right). \tag{2.32}$$

Set $\alpha_1 = 0$ and differentiate this m times with respect to α_2 at $\alpha_2 = 0$ to get

$$\left[\hat{a},\ \hat{a}^{\dagger m}\right] = m\hat{a}^{\dagger m-1}. \tag{2.33}$$

2. On setting $\alpha_1 = \alpha_2 = 0,\ \alpha_3 = 1$, (2.30) yields

$$\exp\left(-\theta\hat{a}^\dagger\hat{a}\right)\hat{a}\exp\left(\theta\hat{a}^\dagger\hat{a}\right) = \exp(\theta)\hat{a}. \tag{2.34}$$

The system of interest in quantum optics described by the harmonic oscillator algebra is the electromagnetic field (see Chap. 6).

2.3.2 The $SU(2)$ Algebra

The Lie algebra generated by the spin operators $\hat{S}_x, \hat{S}_y, \hat{S}_z$ obeying the commutation relations (1.126) is called the $SU(2)$ algebra. SU stands for special unitary. This nomenclature has its origin in the group theory. However, the group theoretic aspects of the algebras are of no relevance to us here. Following the notation of Sect. 1.6., we denote by $\hat{S}_a = \boldsymbol{a} \cdot \hat{\boldsymbol{S}} = a_x \hat{S}_x + a_y \hat{S}_y + a_z \hat{S}_z$ the spin operator along the direction $\boldsymbol{a}$. Use this decomposition and (1.126) to show that the commutation relation between the components in two arbitrary directions $\boldsymbol{a}$ and $\boldsymbol{b}$ is

$$\left[\hat{S}_a,\ \hat{S}_b\right] = \mathrm{i}\,(\boldsymbol{a} \times \boldsymbol{b}) \cdot \hat{\boldsymbol{S}}. \tag{2.35}$$

Now, consider the similarity transformation

$$\hat{S}_a(\theta) = \exp\left(-\mathrm{i}\theta \boldsymbol{n} \cdot \hat{\boldsymbol{S}}\right) \boldsymbol{a} \cdot \hat{\boldsymbol{S}} \exp\left(\mathrm{i}\theta \boldsymbol{n} \cdot \hat{\boldsymbol{S}}\right), \quad \boldsymbol{n} \cdot \boldsymbol{n} = 1. \tag{2.36}$$

Differentiate this with respect to θ and use (2.35) to show that

$$\frac{\mathrm{d}}{\mathrm{d}\theta} \hat{S}_a(\theta) = (\boldsymbol{n} \times \boldsymbol{a}) \cdot \hat{\boldsymbol{S}}(\theta). \tag{2.37}$$

The equation satisfied by $(\boldsymbol{n} \times \boldsymbol{a}) \cdot \boldsymbol{S}(\theta)$, can similarly be found to read

$$\frac{\mathrm{d}}{\mathrm{d}\theta} (\boldsymbol{n} \times \boldsymbol{a}) \cdot \hat{\boldsymbol{S}}(\theta) = \exp\left(-\mathrm{i}\theta \boldsymbol{n} \cdot \hat{\boldsymbol{S}}\right) [\boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{a})] \cdot \hat{\boldsymbol{S}} \exp\left(\mathrm{i}\theta \boldsymbol{n} \cdot \hat{\boldsymbol{S}}\right). \tag{2.38}$$

Now, use the relation $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$ to rewrite (2.38) as

$$\frac{\mathrm{d}}{\mathrm{d}\theta} (\boldsymbol{n} \times \boldsymbol{a}) \cdot \hat{\boldsymbol{S}}(\theta) = -\hat{S}_a(\theta) + (\boldsymbol{a} \cdot \boldsymbol{n})(\boldsymbol{n} \cdot \hat{\boldsymbol{S}}). \tag{2.39}$$

In writing this we have made use of the fact that

$$\exp\left(-\mathrm{i}\theta \boldsymbol{n} \cdot \hat{\boldsymbol{S}}\right) \boldsymbol{n} \cdot \hat{\boldsymbol{S}} \exp\left(\mathrm{i}\theta \boldsymbol{n} \cdot \hat{\boldsymbol{S}}\right) = \boldsymbol{n} \cdot \hat{\boldsymbol{S}} \tag{2.40}$$

is independent of θ. Solution of the coupled set (2.37) and (2.39) yields

$$\begin{aligned} &\exp\left(-\mathrm{i}\theta \boldsymbol{n} \cdot \hat{\boldsymbol{S}}\right) \boldsymbol{a} \cdot \hat{\boldsymbol{S}} \exp\left(\mathrm{i}\theta \boldsymbol{n} \cdot \boldsymbol{S}\right) \\ &= \cos(\theta)\, \boldsymbol{a} \cdot \hat{\boldsymbol{S}} + (\boldsymbol{n} \times \boldsymbol{a}) \cdot \hat{\boldsymbol{S}} \sin(\theta) + [1 - \cos(\theta)] \left(\boldsymbol{n} \cdot \hat{\boldsymbol{S}}\right)\left(\boldsymbol{n} \cdot \boldsymbol{a}\right). \end{aligned} \tag{2.41}$$

If $\boldsymbol{n} \cdot \boldsymbol{a} = 0$ then, also on invoking (2.35), the expression above reduces to

$$\begin{aligned} &\exp\left(-\mathrm{i}\theta \boldsymbol{n} \cdot \hat{\boldsymbol{S}}\right) \boldsymbol{a} \cdot \hat{\boldsymbol{S}} \exp\left(\mathrm{i}\theta \boldsymbol{n} \cdot \boldsymbol{S}\right) \\ &\quad = \cos(\theta)\, \boldsymbol{a} \cdot \hat{\boldsymbol{S}} - \mathrm{i}[\boldsymbol{n} \cdot \hat{\boldsymbol{S}}, \boldsymbol{a} \cdot \hat{\boldsymbol{S}}] \sin(\theta). \end{aligned} \tag{2.42}$$

Some special cases of this are enumerated below.

1. If $\boldsymbol{n} \cdot \hat{\boldsymbol{S}} = \hat{S}_z, \boldsymbol{a} \cdot \hat{\boldsymbol{S}} = \hat{S}_x$ then

$$\exp\left(-\mathrm{i}\theta \hat{S}_z\right) \hat{S}_x \exp\left(\mathrm{i}\theta \hat{S}_x\right) = \cos(\theta)\, \hat{S}_z + \hat{S}_y \sin(\theta). \tag{2.43}$$

2.

$$\hat{S}_{\pm}(\theta) = \exp\left(-\mathrm{i}\theta\hat{S}_z\right)\hat{S}_{\pm}\exp\left(\mathrm{i}\theta\hat{S}_z\right) = \exp(\mp\mathrm{i}\theta)\hat{S}_{\pm}. \tag{2.44}$$

This may be derived by noting that $\mathrm{d}\hat{S}_{\pm}(\theta)/\mathrm{d}\theta = \mp\mathrm{i}\hat{S}_{\pm}(\theta)$.

3.

$$\hat{S}_z(\theta) = \exp\left(-\mathrm{i}\theta\hat{S}_+\right)\hat{S}_z\exp\left(\mathrm{i}\theta\hat{S}_+\right) = \hat{S}_z + \mathrm{i}\theta\hat{S}_+. \tag{2.45}$$

This may be derived by noting that $\mathrm{d}\hat{S}_z(\theta)/\mathrm{d}\theta = \mathrm{i}\hat{S}_+$. Multiply (2.45) on the left by $\exp(\mathrm{i}\theta\hat{S}_+)$, differentiate it m times with respect to θ at $\theta = 0$ and show that

$$[\hat{S}_z, \hat{S}_+^m] = m\hat{S}_+^m \ .$$

4.

$$\hat{S}_-(\theta) = \exp\left(-\mathrm{i}\theta\hat{S}_+\right)\hat{S}_-\exp\left(\mathrm{i}\theta\hat{S}_+\right) = \hat{S}_- - 2\mathrm{i}\theta\hat{S}_z + \theta^2\hat{S}_+. \tag{2.46}$$

This may be derived by showing that $\mathrm{d}\hat{S}_-(\theta)/\mathrm{d}\theta = -2\mathrm{i}\hat{S}_z(\theta)$ where $\hat{S}_z(\theta)$ is as in (2.45). Multiply (2.46) on the left by $\exp(\mathrm{i}\theta\hat{S}_+)$, differentiate it m times with respect to θ at $\theta = 0$ and show that

$$[\hat{S}_-, \hat{S}_+^m] = -2m\hat{S}_+^{m-1}\hat{S}_z - m(m-1)\hat{S}_+^{m-1} \ .$$

The system of interest in quantum optics that is an element of $SU(2)$ is that of a collection of two-level atoms (see Chap. 7).

2.3.3 The $SU(1,1)$ Algebra

The $SU(1,1)$ algebra is generated by hermitian operators $\hat{K}_x$, $\hat{K}_y$, and $\hat{K}_z$ obeying the commutation relations

$$\left[\hat{K}_x,\ \hat{K}_y\right] = -\mathrm{i}\hat{K}_z, \quad \left[\hat{K}_y,\ \hat{K}_z\right] = \mathrm{i}\hat{K}_x, \quad \left[\hat{K}_z,\ \hat{K}_x\right] = \mathrm{i}\hat{K}_y. \tag{2.47}$$

Note that the difference between the commutation relations for $SU(1,1)$ and the relations (1.126) for $SU(2)$ is in the sign of the commutator of the x and y components. Like the case of $SU(2)$ operators, the $SU(1,1)$ operators can also be thought of as components of a vector in a three dimensional space. The space for $SU(1,1)$ operators, however, is not the Euclidean space of $SU(2)$ operators. It is the so called (2+1)-dimensional Minkowski space in which the dot and the cross products between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are defined by

$$\boldsymbol{a}\cdot\boldsymbol{b} = a_x b_x + a_y b_y - a_z b_z, \tag{2.48a}$$

$$(\boldsymbol{a}\times\boldsymbol{b})_i = -\sum_{j,k=x,y,z}\epsilon_{ijk}a_j b_k, \ i \neq z;$$

$$(\boldsymbol{a}\times\boldsymbol{b})_z = \sum_{j,k=x,y,z}\epsilon_{zjk}a_j b_k. \tag{2.48b}$$

Here ϵ_{xyz} is $+1$ (-1) for even (odd) permutation of x, y, z; $\epsilon_{ijk} = 0$ if any two of the i, j, k are same. Note that the scalar product so defined does not have the properties of the Euclidean scalar product as the "norm" $\boldsymbol{a} \cdot \boldsymbol{a}$ of a vector $\boldsymbol{a}$ according to (2.48a) need not be positive. Keeping in mind the definitions (2.48a) and (2.48b) of the dot and the cross products, show that

$$\left[\boldsymbol{a} \cdot \hat{\boldsymbol{K}},\ \boldsymbol{b} \cdot \hat{\boldsymbol{K}}\right] = \mathrm{i}\,(\boldsymbol{a} \times \boldsymbol{b}) \cdot \hat{\boldsymbol{K}}. \tag{2.49}$$

By following the steps outlined above for the case of $SU(2)$, establish also that

$$\begin{aligned}\exp\left(-\mathrm{i}\theta \boldsymbol{n} \cdot \hat{\boldsymbol{K}}\right) \boldsymbol{a} \cdot \hat{\boldsymbol{K}} \exp\left(\mathrm{i}\theta \boldsymbol{n} \cdot \hat{\boldsymbol{K}}\right) \\ = \cosh(\theta) \boldsymbol{a} \cdot \boldsymbol{K} + (\boldsymbol{n} \times \boldsymbol{a}) \sinh(\theta) - [\cosh(\theta) - 1](\boldsymbol{n} \cdot \boldsymbol{a}) \boldsymbol{n} \cdot \hat{\boldsymbol{K}}.\end{aligned} \tag{2.50}$$

It would turn out to be useful to introduce the operators

$$\hat{K}_{\pm} = \hat{K}_x \pm \mathrm{i}\hat{K}_y, \qquad \hat{K}_{+} = \hat{K}_{-}^{\dagger}. \tag{2.51}$$

Apply (2.47) to see that

$$\left[\hat{K}_z,\ \hat{K}_{\pm}\right] = \pm \hat{K}_{\pm}, \qquad \left[\hat{K}_{+},\ \hat{K}_{-}\right] = -2\hat{K}_z. \tag{2.52}$$

Some similarity transforms involving $SU(1,1)$ of frequent occurrence are enumerated below.

1.

$$\hat{K}_{\pm}(\theta) = \exp\left(-\mathrm{i}\theta \hat{K}_z\right) \hat{K}_{\pm} \exp\left(\mathrm{i}\theta \hat{K}_z\right) = \exp(\mp \mathrm{i}\theta) \hat{K}_{\pm}. \tag{2.53}$$

This may be derived by noting that $\mathrm{d}\hat{K}_{\pm}(\theta)/\mathrm{d}\theta = \mp \mathrm{i}\hat{K}_{\pm}(\theta)$.

2.

$$\hat{K}_z(\theta) = \exp\left(-\mathrm{i}\theta \hat{K}_{+}\right) \hat{K}_z \exp\left(\mathrm{i}\theta \hat{K}_{+}\right) = \hat{K}_z + \mathrm{i}\theta \hat{K}_{+}. \tag{2.54}$$

This may be derived by checking that $\mathrm{d}\hat{K}_z(\theta)/\mathrm{d}\theta = \mathrm{i}\hat{K}_{+}$.

3.

$$\hat{K}_{-}(\theta) = \exp\left(-\mathrm{i}\theta \hat{K}_{+}\right) \hat{K}_{-} \exp\left(\mathrm{i}\theta \hat{K}_{+}\right) = \hat{K}_{-} + 2\mathrm{i}\theta \hat{K}_z - \theta^2 \hat{K}_{+}. \tag{2.55}$$

This may be derived by verifying that $\mathrm{d}\hat{K}_{-}(\theta)/\mathrm{d}\theta = 2\mathrm{i}\hat{K}_z(\theta)$ where $\hat{K}_z(\theta)$ is as in (2.54).

The hamiltonian of interest in quantum optics that is an element of $SU(1,1)$ is the one that describes the process of two-photon down-conversion (see Chap. 7).

2.3.4 The $SU(m)$ Algebra

The algebra $SU(2)$ introduced above is a special case of the $SU(m)$ algebra. It is generated by a set of m^2 operators $\{\hat{A}_{ij}\}$, $i,j = 1,\cdots,m$ obeying the commutation relations

$$\left[\hat{A}_{ij},\ \hat{A}_{kl}\right] = \delta_{jk}\hat{A}_{il} - \delta_{il}\hat{A}_{kj}, \quad i,j = 1,2,\cdots,m. \tag{2.56}$$

Verify that the operator

$$\hat{N} = \sum_{i=1}^{m} \hat{A}_{ii} \tag{2.57}$$

commutes with all $\hat{A}_{ij}$. Hence the number of independent generators of $SU(m)$ is $m^2 - 1$. A similarity transformation of a generator by an arbitrary element of the algebra would involve solving a set of up to $m^2 - 1$ coupled equations. However, it will be seen in the next subsection that, by using the so called bosonic representation, the number of coupled equations required to be solved for an arbitrary similarity transformation in $SU(m)$ is m. As a simple example, we consider the similarity transformation generated by a linear combination of the diagonal operators $\hat{A}_{ii}$:

$$\hat{A}_{ij}(\theta) = \exp\left(-\theta\sum_{i=1}^{m}\alpha_i\hat{A}_{ii}\right)\hat{A}_{ij}\exp\left(\theta\sum_{i=1}^{m}\alpha_i\hat{A}_{ii}\right). \tag{2.58}$$

It is easy to verify that $\mathrm{d}\hat{A}_{ij}(\theta)/\mathrm{d}\theta = (\alpha_j - \alpha_i)\hat{A}_{ij}(\theta)$ so that

$$\hat{A}_{ij}(\theta) = \exp\{\theta(\alpha_j - \alpha_i)\}\,\hat{A}_{ij}. \tag{2.59}$$

The hamiltonian of an m-level atom interacting with classical e.m. field and a class of optical parametric processes constitute realizations of $SU(m)$ [30].

2.3.5 The $SU(m,n)$ Algebra

The $SU(2)$, $SU(1,1)$ and $SU(m)$ algebras introduced above are special cases of the $SU(m,n)$ algebra with $SU(m) \equiv SU(m,0)$. The group theoretic aspects of $SU(m,n)$ are of no relevance to us here. For us, it is adequate to introduce $SU(m,n)$ in terms of what is known as its bosonic representation. The $SU(m,n)$ operators in this representation are bilinear combinations of the boson operators $\{\hat{a}_k, \hat{a}_k^\dagger; \hat{b}_p, \hat{b}_p^\dagger\}$ $(k = 1,\ldots,m; p = 1,\ldots,n)$ obeying

$$\left[\hat{a}_k,\ \hat{a}_l^\dagger\right] = \delta_{kl},\ \left[\hat{b}_p,\ \hat{b}_q^\dagger\right] = \delta_{pq},\ \left[\hat{a}_k,\ \hat{b}_p\right] = 0,\ \left[\hat{a}_k,\ \hat{b}_p^\dagger\right] = 0. \tag{2.60}$$

The set of $(m+n)^2$ bilinear combinations of the form

$$\hat{X}_{jk} = \hat{a}_j^\dagger\hat{a}_k, \quad \hat{Y}_{pq} = \hat{b}_p^\dagger\hat{b}_q, \quad \hat{Z}_{jp} = \hat{a}_j\hat{b}_p \tag{2.61}$$

along with their hermitian conjugates generate $SU(m,n)$. These operators commute with the *effective number* operator

$$\hat{N} = \sum_{i=1}^{m} a_i^\dagger a_i - \sum_{p=1}^{n} b_p^\dagger b_p. \tag{2.62}$$

Hence the number of independent generators of $SU(m,n)$ is $(m+n)^2-1$. It should be emphasized that the representation of $SU(m,n)$ in terms of bilinear combinations of bosonic operators does not encompass all its representations.

Consider the similarity transformation

$$\hat{A}(\theta) = \exp\left(-\theta \hat{Z}_{mn}\right) \hat{A} \exp\left(\theta \hat{Z}_{mn}\right). \tag{2.63}$$

Let $\hat{Z}_{mn}$ be an element of $SU(m,n)$ expressed as

$$\hat{Z}_{mn} = \sum_{i,j=1}^{m} \alpha_{ij} \hat{a}_i^\dagger \hat{a}_j + \sum_{p,q=1}^{n} \beta_{pq} \hat{b}_p^\dagger \hat{b}_q + \sum_{i=1}^{m}\sum_{p=1}^{n} \gamma_{ip}^{(1)} \hat{a}_i \hat{b}_p - \sum_{i=1}^{m}\sum_{p=1}^{n} \gamma_{ip}^{(2)} \hat{a}_i^\dagger \hat{b}_p^\dagger. \tag{2.64}$$

Verify invoking the commutation relations that

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\hat{a}_i(\theta) = \sum_{j=1}^{m} \alpha_{ij} \hat{a}_j(\theta) - \sum_{p=1}^{n} \gamma_{ip}^{(2)} \hat{b}_p^\dagger(\theta),$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\hat{b}_p^\dagger(\theta) = -\sum_{q=1}^{n} \beta_{qp} \hat{b}_q^\dagger(\theta) - \sum_{i=1}^{m} \gamma_{ip}^{(1)} \hat{a}_i(\theta). \tag{2.65}$$

These constitute a set of $m+n$ linear coupled equations. Since any element of $SU(m,n)$ is a bilinear combination of $\{\hat{a}_i\}$, $\{\hat{b}_p\}$ and their hermitian conjugates, any $SU(m,n)$ similarity transformation is determined by the solution of $m+n$ coupled linear equations (2.65) and the property (2.17). For a discussion of solution of (2.65) see [30]. In the following we consider its special cases:

1. Verify that

$$\hat{S}_+ = a_1^\dagger \hat{a}_2, \quad \hat{S}_- = a_2^\dagger \hat{a}_1, \quad \hat{S}_z = \frac{1}{2}(a_1^\dagger \hat{a}_1 - a_2^\dagger \hat{a}_2) \tag{2.66}$$

obey the commutation relations (1.128) of $SU(2)$. Consider the similarity transformation

$$\hat{a}_i(\theta) = \exp\left(\mathrm{i}\theta \hat{Z}_2\right) \hat{a}_i \exp\left(-\mathrm{i}\theta \hat{Z}_2\right), \tag{2.67}$$

$$\hat{Z}_2 = \omega_1 \hat{a}_1^\dagger \hat{a}_1 + \omega_2 \hat{a}_2^\dagger \hat{a}_2 + \xi_+ \hat{a}_1^\dagger \hat{a}_2 + \xi_- \hat{a}_2^\dagger \hat{a}_1 \tag{2.68}$$

On differentiating (2.67) with respect to θ we find that $\hat{a}_1(\theta)$ and $\hat{a}_2(\theta)$ obey the coupled equations

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\theta}\hat{a}_1(\theta) = \omega_1\hat{a}_1(\theta) + \xi_+\hat{a}_2(\theta),$$
$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\theta}\hat{a}_2(\theta) = \omega_2\hat{a}_2(\theta) + \xi_-\hat{a}_1(\theta). \tag{2.69}$$

These are easily solvable. Their solution may be used in conjunction with (2.66) and (2.17) to find similarity transformation of any $SU(2)$ operator. In particular, for $\omega_1 = \omega_2 = 0$, (2.69) are solved by

$$\hat{a}_1(\theta) = \cos\left(\sqrt{\xi_+\xi_-}\theta\right) a_1 - \mathrm{i}\sqrt{\frac{\xi_+}{\xi_-}}\sin\left(\sqrt{\xi_+\xi_-}\theta\right) a_2,$$
$$\hat{a}_2(\theta) = \cos\left(\sqrt{\xi_+\xi_-}\theta\right) a_2 - \mathrm{i}\sqrt{\frac{\xi_-}{\xi_+}}\sin\left(\sqrt{\xi_+\xi_-}\theta\right) a_1. \tag{2.70}$$

2. Verify that

$$\hat{K}_+ = a^\dagger\hat{b}^\dagger, \quad \hat{K}_- = \hat{b}\hat{a}, \quad \hat{K}_z = \frac{1}{2}(a^\dagger\hat{a} + b^\dagger\hat{b} + 1) \tag{2.71}$$

obey the commutation relations (2.52) of $SU(1,1)$. Consider the similarity transformation

$$\hat{A}(\theta) = \exp\left(\mathrm{i}\theta\hat{Z}_{11}\right)\hat{A}\exp\left(-\mathrm{i}\theta\hat{Z}_{11}\right), \tag{2.72}$$

$$\hat{Z}_{11} = \omega_a\hat{a}^\dagger\hat{a} + \omega_b\hat{b}^\dagger\hat{b} + \xi_+\hat{a}^\dagger\hat{b}^\dagger + \xi_-\hat{b}\hat{a}. \tag{2.73}$$

It is straightforward to show that

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\theta}\hat{a}(\theta) = \omega_a\hat{a}(\theta) + \xi_+\hat{b}^\dagger(\theta),$$
$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\theta}\hat{b}^\dagger(\theta) = -\omega_b\hat{b}^\dagger(\theta) - \xi_-\hat{a}(\theta). \tag{2.74}$$

These are easily solvable. Their solution may be used in conjunction with (2.71) and (2.17) to find similarity transformation of any $SU(1,1)$ operator. In particular, for $\omega_a = \omega_b = 0$, (2.74) are solved by

$$\hat{a}(\theta) = \cosh\left(\sqrt{\xi_+\xi_-}\theta\right)\hat{a} - \mathrm{i}\sqrt{\frac{\xi_+}{\xi_-}}\sinh\left(\sqrt{\xi_+\xi_-}\theta\right)\hat{b}^\dagger,$$
$$\hat{b}^\dagger(\theta) = \cosh\left(\sqrt{\xi_+\xi_-}\theta\right)\hat{b}^\dagger + \mathrm{i}\sqrt{\frac{\xi_-}{\xi_+}}\sinh\left(\sqrt{\xi_+\xi_-}\theta\right)\hat{a}. \tag{2.75}$$

3. Verify that the $SU(1,1)$ commutation relations (2.52) are obeyed also by the bilinear combinations

$$\hat{K}_+ = \frac{a^{\dagger 2}}{2}, \quad \hat{K}_- = \frac{\hat{a}^2}{2}, \quad \hat{K}_z = \frac{1}{2}(a^\dagger\hat{a} + 1/2) \tag{2.76}$$

of a boson operator with itself. Consider the similarity transformation

$$\hat{A}(\theta) = \exp\left(\mathrm{i}\theta\hat{Z}\right)\hat{A}\exp\left(-\mathrm{i}\theta\hat{Z}\right), \tag{2.77}$$

$$\hat{Z} = \omega_a\hat{a}^\dagger\hat{a} + \frac{\xi_+}{2}\hat{a}^{\dagger 2} + \frac{\xi_-}{2}\hat{a}^2. \tag{2.78}$$

It then follows that

$$\begin{aligned}\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\theta}\hat{a}(\theta) &= \omega_a\hat{a}(\theta) + \xi_+\hat{a}^\dagger(\theta),\\ \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\theta}\hat{a}^\dagger(\theta) &= -\omega_a\hat{a}(\theta) - \xi_-\hat{a}(\theta).\end{aligned} \tag{2.79}$$

These are easily solvable. Their solution for $\omega_a = 0$ reads

$$\hat{a}(\theta) = \cosh\left(\sqrt{\xi_+\xi_-}\theta\right)\hat{a} - \mathrm{i}\sqrt{\frac{\xi_+}{\xi_-}}\sinh\left(\sqrt{\xi_+\xi_-}\theta\right)a^\dagger. \tag{2.80}$$

The hamiltonians of a variety of important optical parametric processes are realizations of the elements of $SU(m,n)$ [30].

2.4 Disentangling an Exponential

By disentangling an exponential we mean expressing the exponential of a sum of operators in terms of a product of the exponentials of operators. If $\hat{A}$ and $\hat{B}$ are given operators then the problem of disentanglement consists in finding operators $\hat{C}_1, \hat{C}_2, \cdots$ such that

$$\exp\left[\hat{A}+\hat{B}\right] = \exp\left(\hat{A}\right)\exp\left(\hat{B}\right)\exp\left(\hat{C}_1\right)\exp\left(\hat{C}_2\right)\ldots \tag{2.81}$$

The $\hat{C}_n$ are combinations of repeated commutator of $\hat{A}$ and $\hat{B}$ (see [31] for details). The expansion (2.81) in general involves an infinite number of $\hat{C}_n's$. Finding analytical expression for the $\hat{C}_n's$ is generally a formidable task. However, as elaborated below, the number of $\hat{C}_n's$ is finite if $\hat{A}$ and $\hat{B}$ are elements of a finite-dimensional Lie algebra.

Consider a Lie algebra generated by $\hat{X}_1, \cdots, \hat{X}_n$. Let

$$\exp\left[\theta\sum_{i=1}^{n}\alpha_i\hat{X}_i\right] = \exp\left[f_1(\theta)\hat{X}_1\right]\cdots\exp\left[f_n(\theta)\hat{X}_n\right]. \tag{2.82}$$

Note that

$$f_i(0) = 0. \tag{2.83}$$

Now, differentiate (2.82) with respect to θ and multiply the resulting expression on the right with the inverse of (2.82) (constructed using (1.26)) to get

$$\sum_{i=1}^{n} \alpha_i \hat{X}_i = \dot{f}_1(\theta)\hat{X}_1 + \dot{f}_2(\theta) \exp\left[f_1(\theta)\hat{X}_1\right] \hat{X}_2 \exp\left[-f_1(\theta)\hat{X}_1\right] + \cdots$$
$$+\dot{f}_n(\theta)\Big\{ \exp\left[f_1(\theta)\hat{X}_1\right] \cdots \exp\left[f_{n-1}(\theta)\hat{X}_{n-1}\right] \hat{X}_n$$
$$\times \exp\left[-f_{n-1}(\theta)\hat{X}_{n-1}\right] \cdots \exp\left[-f_1(\theta)\hat{X}_1\right] \Big\}. \tag{2.84}$$

On carrying the similarity transformations, the right hand side of this equation reduces to a linear combination of the $\hat{X}_i's$. A comparison of the coefficients of the $\hat{X}_i's$ in the resulting equation then leads to differential equations for the $\{f_i(\theta)\}$. The equations so obtained are, in general, non-linear. Their solution determines $\{f_i(\theta)\}$.

We follow the general procedure outlined here to the algebras introduced in previous sections.

2.4.1 The Harmonic Oscillator Algebra

Let $\hat{a}$, $\hat{a}^\dagger$ be the boson operators and let

$$\exp\left[\theta\left\{\alpha_1\hat{a} + \alpha_2\hat{a}^\dagger\hat{a} + \alpha_3\hat{a}^\dagger\right\}\right]$$
$$= \exp\left[f_1(\theta)\hat{a}^\dagger\right] \exp\left[f_2(\theta)\hat{a}^\dagger\hat{a}\right] \exp\left[f_3(\theta)\hat{a}\right] \exp\left[f_4\right]. \tag{2.85}$$

The procedure outlined above alongwith the results of Sect. 2.3.1 yield

$$\dot{f}_1 - f_1\dot{f}_2 = \alpha_3, \quad \dot{f}_2 = \alpha_2, \quad \dot{f}_3 = \alpha_1 \exp(f_2)$$
$$\dot{f}_4 - f_1\dot{f}_3 \exp(-f_2) = 0. \tag{2.86}$$

The solution of these equations reads

$$f_1 = \frac{\alpha_3}{\alpha_2}\left[\exp(\alpha_2\theta) - 1\right], \qquad f_2 = \alpha_2\theta,$$
$$f_3 = \frac{\alpha_1}{\alpha_2}\left[\exp(\alpha_2\theta) - 1\right], \qquad f_4 = \frac{\alpha_1\alpha_3}{\alpha_2^2}\left[\exp(\alpha_2\theta) - \alpha_2\theta - 1\right]. \tag{2.87}$$

In particular, for $\alpha_2 = 0$, $\theta = 1$, it follows by substituting (2.87) in (2.85) that

$$\exp\left[\alpha_1\hat{a} + \alpha_3\hat{a}^\dagger\right] = \exp\left[\alpha_3\hat{a}^\dagger\right] \exp\left[\alpha_1\hat{a}\right] \exp\left[\frac{1}{2}\alpha_1\alpha_3\right]$$
$$= \exp\left[\alpha_1\hat{a}\right] \exp\left[\alpha_3\hat{a}^\dagger\right] \exp\left[-\frac{1}{2}\alpha_1\alpha_3\right]. \tag{2.88}$$

The last line above is obtained by invoking (2.18), (2.17) and (2.31).

In many applications, we need to expand a function $F(\hat{a}^\dagger, \hat{a})$ in powers of $\hat{a}$ and $\hat{a}^\dagger$ such that all the $\hat{a}$ operators lie on the right of all the $\hat{a}^\dagger$ operators. Such an expansion is called *normally ordered* expansion. The expansion of $F(\hat{a}^\dagger, \hat{a})$ is said to be *antinormally ordered* if all the $\hat{a}^\dagger$ operators appear on the right of all the $\hat{a}$ operators. Note that the first line of (2.88) is in normal

ordered form whereas the second line there is its antinormally ordered form. Next we derive those forms for the exponential of $\hat{a}^\dagger\hat{a}$. To that end, let

$$\exp\left[\theta\hat{a}^\dagger\hat{a}\right] = \sum_{m=0}^{\infty} \frac{x^m(\theta)}{m!}\hat{a}^{\dagger m}\hat{a}^m, \tag{2.89}$$

where $x(\theta)$ is an unknown function. To find that function, differentiate (2.89) with respect to θ to get

$$\hat{a}^\dagger\hat{a}\exp\left[\theta\hat{a}^\dagger\hat{a}\right] = \frac{\mathrm{d}x(\theta)}{\mathrm{d}\theta}\sum_{m=0}^{\infty} \frac{x^m(\theta)}{m!}\hat{a}^{\dagger m+1}\hat{a}^{m+1}. \tag{2.90}$$

Now, (i) rewrite the exponential on the left hand side using the series expansion (2.89), (ii) use (2.33) to write $\hat{a}\hat{a}^{\dagger m} = \hat{a}^{\dagger m}\hat{a} + m\hat{a}^{\dagger m-1}$, and (iii) compare the coefficients of $\hat{a}^{\dagger m+1}\hat{a}^{m+1}$ to arrive at

$$\frac{\mathrm{d}x(\theta)}{\mathrm{d}\theta} = x(\theta) + 1. \tag{2.91}$$

Its solution for $x(0) = 0$ reads

$$x(\theta) = \exp(\theta) - 1. \tag{2.92}$$

On substituting this in (2.89) we get

$$\exp\left[\theta\hat{a}^\dagger\hat{a}\right] = \sum_{m=0}^{\infty} \frac{(\exp(\theta)-1)^m}{m!}\hat{a}^{\dagger m}\hat{a}^m. \tag{2.93}$$

The antinormal form of the exponential of $\hat{a}^\dagger\hat{a}$ may be derived by starting with the expansion

$$\exp\left[\theta\hat{a}\hat{a}^\dagger\right] = \sum_{m=0}^{\infty} \frac{y^m(\theta)}{m!}\hat{a}^m\hat{a}^{\dagger m}. \tag{2.94}$$

Follow the steps outlined above for determining the normal ordered form of the exponential of $\hat{a}^\dagger\hat{a}$ and show that

$$\exp\left[\theta\hat{a}^\dagger\hat{a}\right] = \exp(-\theta)\sum_{m=0}^{\infty} \frac{(1-\exp(-\theta))^m}{m!}\hat{a}^m\hat{a}^{\dagger m}. \tag{2.95}$$

2.4.2 The $SU(2)$ Algebra

Let the exponential of an element of $SU(2)$ be disentangled in the form

$$\begin{aligned}
&\exp\left[\mathrm{i}\theta\left\{\alpha_+\hat{S}_+ + \alpha_z\hat{S}_z + \alpha_-\hat{S}_-\right\}\right] \\
&= \exp\left(f_+(\theta)\hat{S}_+\right)\exp\left(f_z(\theta)\hat{S}_z\right)\exp\left(f_-(\theta)\hat{S}_-\right) \\
&= \exp\left(\phi_-(\theta)\hat{S}_-\right)\exp\left(-\phi_z(\theta)\hat{S}_z\right)\exp\left(\phi_+(\theta)\hat{S}_+\right).
\end{aligned} \tag{2.96}$$

We outline below a derivation of the first equation above. The second equation follows by noting that the disentangling relation is a consequence of only the commutation relations and that if $\hat{S}_\pm \to \hat{S}_\mp$ then the commutation relations of $(\hat{S}_\mp, -\hat{S}_z)$ are the same as those of $(\hat{S}_\pm, \hat{S}_z)$. Hence $\phi_{\pm,z}(\alpha_+, \alpha_-, \alpha_z) = f_{\mp,z}(\alpha_-, \alpha_+, -\alpha_z)$.

By following the procedure outlined in the beginning of this section we obtain non-linear differential equations

$$\dot{f}_+ - f_+\dot{f}_z - f_+^2\dot{f}_- \exp[-f_z] = \mathrm{i}\alpha_+, \tag{2.97a}$$

$$\dot{f}_z + 2f_+\dot{f}_- \exp[-f_z] = \mathrm{i}\alpha_z, \tag{2.97b}$$

$$\dot{f}_- \exp[-f_z] = \mathrm{i}\alpha_-. \tag{2.97c}$$

Substitution of (2.97c) in (2.97a) and (2.97b) gives

$$\dot{f}_+ - f_+\dot{f}_z - \mathrm{i}f_+^2\alpha_- = \mathrm{i}\alpha_+, \tag{2.98a}$$

$$\dot{f}_z + 2\mathrm{i}f_+\alpha_- = \mathrm{i}\alpha_z. \tag{2.98b}$$

Elimination of $\dot{f}_z$ between these equations leads to the Riccati equation

$$\mathrm{i}\dot{f}_+ + \alpha_z f_+ - \alpha_- f_+^2 + \alpha_+ = 0. \tag{2.99}$$

Follow the standard method for solving a Riccati equation or verify by direct substitution that (2.99), with $f_+(0) = 0$, is solved by

$$f_+ = \frac{\mathrm{i}\alpha_+}{\Gamma_1} \frac{\sin(\Gamma_1\theta)}{\cos(\Gamma_1\theta) - \mathrm{i}\alpha_z \sin(\Gamma_1\theta)/2\Gamma_1}, \tag{2.100}$$

$$\Gamma_1^2 = \alpha_+\alpha_- + \frac{\alpha_z^2}{4}. \tag{2.101}$$

The function f_z can now be determined by combining (2.100) and (2.98b). The solution of the resulting equation yields

$$f_z = -2\ln\left[\cos(\Gamma_1\theta) - \mathrm{i}\frac{\alpha_z}{2\Gamma_1}\sin(\Gamma_1\theta)\right]. \tag{2.102}$$

The expression for f_-, obtained by combining (2.102) and (2.97c), reads

$$f_- = \frac{\mathrm{i}\alpha_-}{\Gamma_1} \frac{\sin(\Gamma_1\theta)}{\cos(\Gamma_1\theta) - \mathrm{i}\alpha_z \sin(\Gamma_1\theta)/2\Gamma_1}. \tag{2.103}$$

2.4.3 $SU(1,1)$ Algebra

Let the exponential of an element of $SU(1,1)$ be disentangled in the form

$$\begin{aligned} &\exp\left[\theta\left\{\alpha_+\hat{K}_+ + \alpha_z\hat{K}_z + \alpha_-\hat{K}_-\right\}\right] \\ &= \exp[\phi_+(\theta)\hat{K}_+]\exp[\phi_z(\theta)\hat{K}_z]\exp[\phi_-(\theta)\hat{K}_-]. \end{aligned} \tag{2.104}$$

The procedure outlined in the beginning of this section leads to following non-linear differential equations for unknown functions $\phi_\pm(\theta), \phi_z(\theta)$:

$$\begin{aligned}\dot{\phi}_+ - \phi_+\dot{\phi}_z + \phi_+^2\dot{\phi}_- \exp[-\phi_z] &= \alpha_+,\\ \dot{\phi}_z - 2\phi_+\dot{\phi}_- \exp[-\phi_z] &= \alpha_z,\\ \dot{\phi}_- \exp[-\phi_z] &= \alpha_-.\end{aligned} \tag{2.105}$$

Following the method outlined in the last subsection we find that

$$\dot{\phi}_+ - \phi_+^2\alpha_- - \phi_+\alpha_z - \alpha_+ = 0. \tag{2.106}$$

The solution of this equation and its use in solving the equations for ϕ_z and ϕ_- yields

$$\begin{aligned}\phi_+ &= \frac{\alpha_+}{\Gamma_2}\frac{\sinh(\Gamma_2\theta)}{\cosh(\Gamma_2\theta) - \alpha_z\sinh(\Gamma_2\theta)/2\Gamma_2},\\ \phi_z &= -2\ln\left[\cosh(\Gamma_2\theta) - \frac{\alpha_z}{2\Gamma_2}\sinh(\Gamma_2\theta)\right],\\ \phi_- &= \frac{\alpha_-}{\Gamma_2}\frac{\sinh(\Gamma_2\theta)}{\cosh(\Gamma_2\theta) - \alpha_z\sinh(\Gamma_2\theta)/2\Gamma_2},\end{aligned} \tag{2.107}$$

$$\Gamma_2^2 = \frac{\alpha_z^2}{4} - \alpha_+\alpha_-. \tag{2.108}$$

2.5 Time-Ordered Exponential Integral

We have seen in the last chapter that the study of the motion of a quantum system governed by a time-dependent hamiltonian requires evaluation of a time-ordered exponential integral. If $\hat{X}_1, \cdots, \hat{X}_n$ generate a Lie algebra then we may write

$$\begin{aligned}&\overleftarrow{T}\exp\left[\int_0^t \mathrm{d}\tau \sum_{i=1}^n \alpha_i(\tau)\hat{X}_i\right]\\ &= \exp\left(f_1(t)\hat{X}_1\right)\cdots\exp\left(f_n(t)\hat{X}_n\right).\end{aligned} \tag{2.109}$$

On differentiating this with respect to t and on multiplying the resulting expression on the right by the inverse of (2.109) (constructed using (1.26)) we arrive essentially at the equation (2.84) (with θ there identified for the present as t) except that whereas the the $\alpha_i's$ appearing there are independent of the differentiation parameter θ, they are now dependent on the differentiation parameter t. Explicit results for some algebras of interest are derived below.

2.5.1 Harmonic Oscillator Algebra

In this case, let

$$\overleftarrow{T}\exp\left[\int_0^t \mathrm{d}\tau\left\{\alpha_1(\tau)\hat{a}+\alpha_2(\tau)\hat{a}^\dagger\hat{a}+\alpha_3(\tau)\hat{a}^\dagger\right\}\right]$$
$$=\exp\left[f_1(t)\hat{a}^\dagger\right]\exp\left[f_2(t)\hat{a}^\dagger\hat{a}\right]\exp\left[f_3(t)\hat{a}\right]\exp[f_4(t)]. \tag{2.110}$$

As argued above, the $f_i(t)$'s are determined by solving (2.86) but the $\alpha_i(t)$'s appearing there are now time-dependent. The solution of those equations is, therefore, no longer given by (2.87). It may be shown that the solution of (2.86) for time-dependent $\alpha_i(t)$'s is

$$f_1(t)=\int_0^t \mathrm{d}\tau'\exp\Big[\int_{\tau'}^t \alpha_2(\tau)\mathrm{d}\tau\Big]\alpha_3(\tau'),\quad f_2(t)=\int_0^t \alpha_2(\tau)\mathrm{d}\tau,$$
$$f_3(t)=\int_0^t \alpha_1(\tau)\exp[f_2(\tau)]\mathrm{d}\tau,\quad f_4(t)=\int_0^t f_1(\tau)\alpha_1(\tau)\mathrm{d}\tau. \tag{2.111}$$

These may be evaluated for the $\alpha(t)'s$ for the problem at hand. The expressions (2.111), of course, reduce to (2.87) if the $\alpha_i's$ are time-independent.

2.5.2 $SU(2)$ Algebra

Let the time-ordered integral involving an element of $SU(2)$ be expressed as

$$\overleftarrow{T}\exp\left[\mathrm{i}\int_0^t \mathrm{d}\tau\Big\{\alpha_+(\tau)\hat{S}_+ +\alpha_z(\tau)\hat{S}_z+\alpha_-(\tau)\hat{S}_-\Big\}\right]$$
$$=\exp\left[f_+(t)\hat{S}_+\right]\exp\left[f_z(t)\hat{S}_z\right]\exp\left[f_-(t)\hat{S}_-\right]. \tag{2.112}$$

The equations obeyed by $f_\pm(t)$ and $f_z(t)$ are the same as (2.97a)-(2.97c) but with time-dependent $\alpha_i(t)'s$. However, note that the steps leading to (2.99) are not influenced by the time-dependence of the $\alpha's$. Hence f_+ even in the present circumstance obeys (2.99) but now with time-dependent coefficients. That equation can not be solved in general. It can be handled analytically or numerically depending on the $\alpha(t)'s$ for the problem at hand.

2.5.3 The $SU(1,1)$ Algebra

Let the time-ordered integral of the exponential of an element of $SU(1,1)$ be expressed as

$$\overleftarrow{T}\exp\left[\mathrm{i}\int_0^t \mathrm{d}\tau\Big\{\alpha_+(\tau)\hat{K}_+ +\alpha_z(\tau)\hat{K}_z+\alpha_-(t)\hat{K}_-\Big\}\right]$$
$$=\exp\left[\phi_+(t)\hat{K}_+\right]\exp\left[\phi_z(t)\hat{K}_z\right]\exp\left[\phi_-(t)\hat{K}_-\right]. \tag{2.113}$$

The $\phi_\pm(t),\phi_z(t)$ obey (2.105) but the $\alpha_i(t)'s$ are now time-dependent. The ϕ_+, however, still obeys (2.106) which can not be solved analytically for general time-dependence of the $\alpha_i(t)'s$. It can be handled analytically or numerically depending on the $\alpha(t)'s$ for the problem at hand.

3. Representations of Some Lie Algebras

Comparison of the quantum theory with experiments is made by evaluating expectation values of observables. Evaluation of expectation values is carried in a c-number representation of the vector space of the states of the system by choosing a suitable basis. A basis that immediately suggests itself for investigating time-evolution of a system is the one spanned by the eigenvectors of the hamiltonian governing its motion. As has been pointed out in the last chapter, hamiltonians encountered frequently in quantum optics can be classified as elements of the Lie algebras or of their direct products. Hence, the problem of representation of a quantum optical system in terms of the eigenvectors of its hamiltonian reduces to that of finding eigenvectors of hermitian elements of the Lie algebras introduced in the last chapter. However, it will be seen that there are other bases which prove to be of not only mathematical interest but also of importance in understanding various physics aspects.

In this chapter we address the question of constructing representations of some Lie algebras of interest in quantum optics.

3.1 Representation by Eigenvectors and Group Parameters

The bases of a Lie algebra may be classified as (i) the bases constituted by the eigenvectors of the elements of the algebra, (ii) the bases labeled by the parameters characterizing the associated group to be defined in the sequel.

3.1.1 Bases Constituted by Eigenvectors

The most convenient bases are, of course, the ones formed by complete sets of orthonormal vectors. Since the eigenvectors of a hermitian operator constitute a complete orthonormal set of vectors, a way of constructing an orthonormal basis of an algebra is in terms of the eigenvectors of a hermitian generator of the algebra. Consider the eigenvalue equation

$$\hat{X}|\lambda\rangle = \lambda|\lambda\rangle, \quad \hat{X} = \hat{X}^\dagger \tag{3.1}$$

for a hermitian element $\hat{X}$ in the algebra. The set of its eigenvectors $\{|\lambda\rangle\}$ constitutes an orthonormal basis. We can construct different eigenbases by

solving (3.1) for different $\hat{X}$ in the algebra. However, not all the sets so obtained are inequivalent in the sense discussed below.

On operating (3.1) with a unitary operator $\hat{U}$ it follows that

$$\left(\hat{U}\hat{X}\hat{U}^{\dagger}\right)\hat{U}|\lambda\rangle = \lambda\hat{U}|\lambda\rangle. \tag{3.2}$$

From this we infer that $\hat{U}|\lambda\rangle$ is an eigenvector of the hermitian operator $\hat{U}\hat{X}\hat{U}^{\dagger}$ with the same eigenvalue λ. Now, let $\hat{U}$ be given by

$$\hat{U}(\{\alpha\}) = \exp\left\{\mathrm{i}\sum_{j=1}^{n}\alpha_j\hat{X}_j\right\}, \quad \hat{X}_j = \hat{X}_j^{\dagger}, \tag{3.3}$$

where $\{\alpha\} \equiv (\alpha_1, \ldots, \alpha_n)$ is a set of real constants and $\{\hat{X}_j\}$ are the generators of the given algebra. It is not difficult to show that the set of all $\hat{U}(\{\alpha\})$, parameterized by the set of real numbers $\{\alpha\}$, constitute a group. It is called the *Lie group* associated with the given Lie algebra. In particular, we note the group property

$$\hat{U}\Big(\{\alpha\}\Big)\hat{U}\Big(\{\alpha'\}\Big) = \hat{U}\Big(\{\alpha^{"}(\alpha,\alpha')\}\Big) \tag{3.4}$$

where $\alpha^{"}(\alpha,\alpha')$ is a function of α and α'. The relation (3.4) states that the product of two group elements is also an element of the group.

Let us now reexamine (3.1) and (3.2). The operator $\hat{U}(\{\alpha\})\hat{X}\hat{U}^{\dagger}(\{\alpha\})$ in (3.2), being a similarity transform of $\hat{X}$ belongs, as discussed in Chap. 2, to the algebra of $\hat{X}$. The operators $\hat{X}$ and $\hat{U}(\{\alpha\})\hat{X}\hat{U}^{\dagger}(\{\alpha\})$, related by a unitary transformation in the associated group, are said to be *unitarily equivalent*. Their corresponding eigenstates $\{|\lambda\rangle\}$ and $\{\hat{U}|\lambda\rangle\}$ constitute *unitarily equivalent representations*. Unitarily inequivalent bases are constructed by solving the eigenvalue problem for unitarily inequivalent generators of the algebra.

Another instructive way of representing an algebra, discussed next, is in terms of the bases labeled by the group parameters $\{\alpha\}$.

3.1.2 Bases Labeled by Group Parameters

Let $|\psi_0\rangle$ be a vector in the space in which the elements of the Lie algebra generated by $\{\hat{X}_i\}$ act. We show that the states

$$|\psi_0(\{\alpha\})\rangle = \exp\left[\mathrm{i}\sum_{j=1}^{n}\alpha_j\hat{X}_j\right]|\psi_0\rangle \equiv \hat{U}(\{\alpha\})|\psi_0\rangle, \quad \hat{X}_j = \hat{X}_j^{\dagger}, \tag{3.5}$$

constitute a complete set as a function of $\{\alpha\}$. To that end, consider the operator

$$\hat{A} = \int \mathrm{d}\mu(\{\alpha\})|\psi_0(\{\alpha\})\rangle\langle\psi_0(\{\alpha\})| \tag{3.6}$$

Here $d\mu(\{\alpha\})$ is an invariant measure on the parameter space of the group [32]. Now, (i) rewrite $|\psi_0(\{\alpha\})\rangle$ in (3.6) as in (3.5), (ii) carry the similarity transformation of the resulting expression by $\hat{U}(\{\alpha'\})$, and (iii) use (3.4) to arrive at

$$\hat{U}\Big(\{\alpha'\}\Big)\hat{A}\hat{U}^{\dagger}\Big(\{\alpha'\}\Big) = \int d\mu(\{\alpha\})\Big[\hat{U}\Big(\{\alpha''(\{\alpha\},\{\alpha'\})\Big) \times|\psi(0)\rangle\langle\psi(0)|\hat{U}^{\dagger}\Big(\{\alpha''(\{\alpha\},\{\alpha'\})\Big)\Big]. \tag{3.7}$$

Next, transform the variables $\{\alpha\}$ of integration in this to new variables $\{\alpha''\}$. On using the fact that the measure is an invariant of the group, i.e. $d\mu(\{\alpha\}) = d\mu(\{\alpha''\})$, the integral in (3.7) reduces to the one in (3.6), so that

$$\hat{U}(\{\alpha\})\hat{A}U^{\dagger}(\{\alpha\}) = \hat{A}, \tag{3.8}$$

i.e. $\Big[\hat{U}(\{\alpha\}),\ \hat{A}\Big] = 0$. Since $\hat{U}(\{\alpha\})$ is an arbitrary element of an irreducible representation of the group, it follows from the Schur's Lemma that $\hat{A}$ must be a constant multiple of unity, i.e. $\hat{A} = cI$ where c is a constant to be determined. This constant is fixed by taking the expectation value of (3.6) (with $\hat{A} = cI$) in a state $|\Psi\rangle$ so that

$$c = \int d\mu(\{\alpha\})\Big|\langle\Psi|\psi_0\{\alpha\}\rangle\Big|^2. \tag{3.9}$$

On redefining the measure by suitably scaling it by the constant c, emerges the completeness relation

$$\int d\mu(\{\alpha\})|\psi_0\{\alpha\}\rangle\langle\psi_0\{\alpha\}| = I, \tag{3.10}$$

for the states $|\psi_0\{\alpha\}\rangle$ parameterized by $\{\alpha\}$.

However, not all the $\alpha's$ need be essential for characterizing the state $|\psi_0\{\alpha\}\rangle$. For, if the fiducial state $|\psi_0\rangle$ is an eigenstate of some of the generators, say, $\hat{X}_m, \hat{X}_{m+1}, \ldots$ then those generators must commute and thus form a subalgebra, say, h. The group generated by the operators in h is called the *stability group* or *stationary group* of $|\psi_0\rangle$. The transformation $\hat{U}(\{\alpha\})$ can then be written as a product of the stability transformation and the transformation generated by the elements not belonging to h. The stability transformation, acting on $|\psi_0\rangle$, gives rise to only a phase factor. Hence the variables $\alpha_m, \alpha_{m+1}..$ associated with those transformations do not play any essential role. The states (3.5), without those operators in $\hat{U}(\{\alpha\})$ of which $|\psi_0\rangle$ is an eigenstate, are called the *Perelomov coherent states* or the *generalized coherent states* [33]. Note also that, by virtue of the group property (3.4), the action of a group element on the coherent state results in another coherent state.

Using the resolution of identity, (3.10), any state $|\Psi\rangle$ can be expanded in terms of the generalized coherent states as

$$|\Psi\rangle = \int \mathrm{d}\mu(\{\alpha\})|\psi_0(\{\alpha\})\rangle\langle\psi_0(\{\alpha\})|\Psi\rangle. \tag{3.11}$$

The function $\langle\psi_0(\{\alpha\})|\Psi\rangle$ provides a representation of the state $|\Psi\rangle$ in terms of the generalized coherent states. The set of generalized coherent states is, in fact, overcomplete in the sense that it contains subsets which are complete (see [34] for details).

Perelomov's definition generalizes the concept of coherent states. The concept that it seeks to generalize is, in fact, of the Glauber coherent states (see Sect. 3.2) introduced by Glauber [35] as those states of the electromagnetic field which give maximum contrast in a two-slit interference pattern (see Chap. 6). These states and their generalizations have since played a significant role in diverse fields [34, 36]. The guiding characteristics for identifying similar states of other systems are that (i) the Glauber coherent state $|\alpha\rangle$ is labeled by a continuously varying parameter α, (ii) the scalar product $\langle\beta|\alpha\rangle$ is continuous as a function of the labels of the states, (iii) the set of states $|\alpha\rangle$ is complete, (iv) $|\alpha\rangle$ is an eigenstate of the harmonic oscillator (h.o.) annihilation operator (see (3.24)), (v) it is generated by the h.o. unitary group transformation on the vacuum state (see (3.27)) and, (vi) it is an uncorrelated equal variance minimum uncertainty state (UEVMUS)(as defined circa (1.69)) of $(\hat{q}, \hat{p})$.

The features (i)-(iii) characterize a coherent state of any system. Accordingly, minimum requirements for a state $|\{z\}\rangle$ of any system to be called a coherent state are that $\{z\}$ be a set of continuously varying parameter; $\langle\{z'\}|\{z\}\rangle$ be continuous as a function of the labels of the states and; the set of states $|\{z\}\rangle$ as a function of $\{z\}$ should be complete. While there may be more than one set of states with these features, the tag 'coherent' can be pinned only to a set possessing some specified properties. For example, as we will see in Sect. 3.2, there are more than one complete sets of continuously labeled h.o. states but the name coherent is given to the set of states in which the two-slit interference pattern exhibits maximum contrast. However, in the absence of a knowledge of any similar property for other systems, the properties defining the coherent states has been a debatable issue. In the case of systems describable by the hamiltonians belonging to one or the other Lie algebra, the definition of coherent states is extended by generalizing to other systems one or the other methods (iv)-(vi) of construction of the Glauber coherent states. Those methods, though equivalent for constructing h.o. coherent states, are not so for other systems. For an extension of the concept of coherent states to a general one-dimensional potential, see [37]. In the following we restrict our attention to the coherent states of the Lie groups.

The generalization based on (iv) consists in defining a coherent state as an eigenstate of an annihilation operator of the algebra. This approach, however, is restrictive as not all the algebras contain an operator whose eigenvalue problem is solved by continuously labeled states. As an example, note that the only solution of the eigenvalue equation $\hat{S}_-|\psi\rangle = 0$ of the $SU(2)$ algebra is

$|\psi\rangle = |S, -S\rangle$ (see Sect. 3.3 for the notation). On the other hand, as we will see in Sect. 3.4, the $SU(1,1)$ annihilation operator $\hat{K}_-$ does admit continuously labeled eigenstates. However, as discussed in that section, those solutions do not possess analog of the group property (v).

Perelomov's definition of a coherent state, as given above, is clearly a generalization of the property (v). It holds for any algebra but ignores (vi).

The property (vi) contains in it the essential physics of the coherent states. For, the notion of minimum uncertainty states is associated with the states in which a system behaves closest to its classical counterpart. Search for such states dates back to the early days of quantum mechanics. Schrödinger [38] labeled the states whose wave-packets do not spread in time under the h.o. potential as the most classical states of the h.o. Those are nothing but the Glauber coherent states. The coherent states of the systems described by the Lie algebras and possessing the property analogous to (vi) may be constructed by restricting the choice of the fiducial state in Perelomov's definition to minimum uncertainty states of the operators in the algebra in a way described below (see also [39]-[41]). This approach unifies the properties (iv)-(vi) of the h.o. coherent states.

In view of preceding delibrationns, let $|\psi_0\rangle$ in (3.5) be a minimum uncertainty state of a pair of non-commuting hermitian generators, say, $\hat{X}_1$, $\hat{X}_2$ in the given algebra. Recall from (1.66) that a minimum uncertainty state $|\psi_0\rangle$ of $\hat{X}_1$, $\hat{X}_2$ solves

$$\left[\hat{X}_1 + \mathrm{i}\lambda\hat{X}_2\right]|\psi_0\rangle = z|\psi_0\rangle. \tag{3.12}$$

It is straightforward to verify by operating (3.12) by $\hat{U}(\{\alpha\})$ that the generalized coherent state (3.5) is a minimum uncertainty state of the pair $\hat{U}(\{\alpha\})\hat{X}_1\hat{U}^\dagger(\{\alpha\})$, $\hat{U}(\{\alpha\})\hat{X}_2\hat{U}^\dagger(\{\alpha\})$. The coherent states (3.5) obtained with $|\psi_0\rangle$ as a solution of (3.12) may, therefore, be called *minimum uncertainty coherent states* (MUCS). Recall from discussion circa (1.69) that if $\lambda = \pm 1$ then $\hat{X}_1$, $\hat{X}_2$ are uncorrelated and their variances ΔX_1^2 and ΔX_2^2 are equal. The MUCS corresponding to $\lambda = \pm 1$ may, therefore, be named *uncorrelated equal variance minimum uncertainty coherent states* (UEVMUCS).

We can thus construct different sets of MUCS by choosing the minimum uncertainty states corresponding to different pairs of non-commuting generators as the fiducial states. However, not all such sets are unitarily inequivalent. For, as remarked above, if $|\psi_0\rangle$ is a minimum uncertainty state of the pair $\left(\hat{X}_1,\ \hat{X}_2\right)$ then $\hat{U}|\psi_0\rangle$ is a minimum uncertainty state of the pair $\left(\hat{U}\hat{X}_1\hat{U}^\dagger,\ \hat{U}\hat{X}_2\hat{U}^\dagger\right)$. Thus the set of MUCS obtained by choosing the minimum uncertainty states of a pair of generators $\left(\hat{X}_1,\ \hat{X}_2\right)$ as the fiducial state is unitarily equivalent to that constructed by choosing the minimum uncertainty states of the pair of generators $\left(\hat{U}\hat{X}_1\hat{U}^\dagger,\ \hat{U}\hat{X}_2\hat{U}^\dagger\right)$ as the fiducial state. Hence, a pair of generators $\left(\hat{X}_1,\ \hat{X}_2\right)$ is said to be unitarily equivalent

to another pair $\left(\hat{X}_i,\ \hat{X}_j\right)$ if there exists a unitary transformation $\hat{U}$ in the associated group such that $\left(\hat{X}_1,\ \hat{X}_2\right) = \left(\hat{U}\hat{X}_i\hat{U}^\dagger,\ \hat{U}\hat{X}_j\hat{U}^\dagger\right)$, i.e. if $\hat{X}_i$ and $\hat{X}_j$ are unitarily equivalent, respectively, to $\hat{X}_1$ and $\hat{X}_2$ by the same unitary transformation in the associated group. A complete set of MUCS of a given algebra consists of all unitarily inequivalent MUCS.

In the following we construct inequivalent representations of some algebras of interest in quantum optics.

3.2 Representations of Harmonic Oscillator Algebra

Recall from Chap. 2 that the harmonic oscillator algebra is generated by the operators $\{\hat{a}, \hat{a}^\dagger, \hat{a}^\dagger\hat{a}, I\}$ obeying the commutation relations (2.26). Let

$$\hat{q} = \frac{1}{\sqrt{2}}\left(\hat{a} + \hat{a}^\dagger\right), \quad \hat{p} = \frac{\mathrm{i}}{\sqrt{2}}\left(\hat{a}^\dagger - \hat{a}\right), \quad [\hat{q}, \hat{p}] = \mathrm{i}. \tag{3.13}$$

The set $(\hat{q}, \hat{p}, \hat{a}^\dagger\hat{a}, I)$ constitutes hermitian generators of the h.o. algebra. Now, by inspecting (2.30) it may be verified that there does not exist any h.o. group transformation that can transform any linear combination of $(\hat{q}, \hat{p})$ to $\hat{a}^\dagger\hat{a}$. The set $(\hat{q}, \hat{p})$ and the operator $\hat{a}^\dagger\hat{a}$ are therefore unitarily inequivalent. Bearing this in mind we construct below the orthonormal bases and the MUCS of the harmonic oscillator algebra.

3.2.1 Orthonormal Bases

As a consequence of the observation above, this algebra admits two sets of unitarily inequivalent bases: one formed by the eigenstates of $\hat{a}^\dagger\hat{a}$ and the other by the eigenstates of $\hat{q}$ or of $\hat{p}$.

Eigenstates of $\hat{a}^\dagger\hat{a}$. Recall that the eigenstates $|n\rangle$ of $\hat{a}^\dagger\hat{a}$, called the *number or Fock states*, are such that

$$\hat{a}^\dagger\hat{a}|n\rangle = n|n\rangle, \qquad n = 0, 1, 2, \ldots \tag{3.14}$$

The completeness and orthonormality relations for these states are

$$\sum_{n=0}^{\infty} |n\rangle\langle n| = I, \qquad \langle m|n\rangle = \delta_{mn}. \tag{3.15}$$

Using the commutation relations, it is straightforward to show that

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle, \qquad \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle. \tag{3.16}$$

These, on repeated application, yield

$$\hat{a}^m|n\rangle = \sqrt{\frac{n!}{(n-m)!}}|n-m\rangle, \qquad \hat{a}^{\dagger m}|n\rangle = \sqrt{\frac{(m+n)!}{n!}}|n+m\rangle. \tag{3.17}$$

The operator $\hat{a}$ ($\hat{a}^\dagger$) thus lowers (raises) the number state and is, therefore, called *lowering or annihilation (raising or creation) operator*. Note that

$$\hat{a}|0\rangle = 0. \tag{3.18}$$

The state $|0\rangle$ is called the *vacuum* state.

In applications, it is at times useful to know the expression of the product $|m\rangle\langle n|$ in terms of the h.o. operators. We derive the desired expression first for $|0\rangle\langle 0|$. To that end, invoke (1.35) to note that

$$\begin{aligned}\lim_{\beta\to\infty} \exp\left(-\beta\hat{a}^\dagger\hat{a}\right) &= \lim_{\beta\to\infty} \sum_{m=0}^{\infty} \exp(-\beta m)|m\rangle\langle m| \\ &= |0\rangle\langle 0|. \end{aligned} \tag{3.19}$$

Now, express the exponential operator above in the normal ordered form using (2.93) so that

$$\begin{aligned}|0\rangle\langle 0| &= \lim_{\beta\to\infty} \sum_{m=0}^{\infty} \frac{(-)^m}{m!} \left(1 - \exp(-\beta)\right)^m \hat{a}^{\dagger m}\hat{a}^m \\ &= \sum_{m=0}^{\infty} \frac{(-)^m}{m!} \hat{a}^{\dagger m}\hat{a}^m. \end{aligned} \tag{3.20}$$

This expresses $|0\rangle\langle 0|$ in terms of normally-ordered h.o. operators. Now, (3.17) implies that

$$|m\rangle\langle n| = \frac{1}{\sqrt{m!n!}} \hat{a}^{\dagger m}|0\rangle\langle 0|\hat{a}^n. \tag{3.21}$$

Combination of this with (3.20) expresses $|m\rangle\langle n|$ in terms of normally ordered product of the h.o. operators.

Eigenstates of $\hat{q}$. The other set of unitarily inequivalent orthonormal basis is formed, as stated above, by the eigenstates of $\hat{q}$ or of $\hat{p}$. The eigenstates of $\hat{q}$ follow as a special case of the eigenvalue equation (3.42) solved in the next subsection.

3.2.2 Minimum Uncertainty Coherent States

The inequivalent pairs of generators of inequivalent sets of minimum uncertainty states of harmonic oscillator are evidently $(\hat{q}, \hat{p})$, and $(\hat{q}, \hat{a}^\dagger\hat{a})$ (or $(\hat{p}, \hat{a}^\dagger\hat{a})$). The corresponding minimum uncertainty states are constructed as follows.

MUCS for the Pair $(\hat{q}, \hat{p})$. The MUCS for the pair $(\hat{q}, \hat{p})$ are given by substituting for $|\psi\rangle$ in

$$|\{\beta\}\rangle = \exp\left[-\mathrm{i}\left\{\beta_1\hat{a}^\dagger\hat{a} + \beta_2^*\hat{a} + \beta_2\hat{a}^\dagger\right\}\right]|\psi\rangle, \ (\beta_1 \text{ real}) \tag{3.22}$$

the solution $|\lambda, \alpha\rangle$ of the equation for the MUS of $\hat{q}, \hat{p}$:

$$[\hat{q} + i\lambda\hat{p}]\,|\lambda, \alpha\rangle = \alpha\sqrt{2}|\lambda, \alpha\rangle \tag{3.23}$$

and by dropping from the exponent in (3.22) the operators of which $|\lambda, \alpha\rangle$ is an eigenstate. We construct first the UEVMUS corresponding to $\lambda = 1$. On invoking the definition (3.13), (3.23) for $\lambda = 1$ reduces to the eigenvalue equation ($|\alpha\rangle \equiv |1, \alpha\rangle$)

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \tag{3.24}$$

for the h.o. annihilation operator. Using (2.31), this may be rewritten as

$$\hat{D}(\alpha)\hat{a}\hat{D}^{\dagger}(\alpha)|\alpha\rangle = 0. \tag{3.25}$$

where the unitary *displacement operator* $\hat{D}(\alpha)$ is defined by

$$\hat{D}(\alpha) = \exp(\alpha\hat{a}^{\dagger} - \alpha^{*}\hat{a}). \tag{3.26}$$

On comparing (3.25) with (3.18), it follows that $\hat{D}^{\dagger}(\alpha)|\alpha\rangle = |0\rangle$ i.e.

$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle. \tag{3.27}$$

Note that $|\alpha\rangle$ has the structure of a Perelomov coherent state, for, it is expressible as a result of transformation of a state, in this case the vacuum state $|0\rangle$, by the operator $\hat{D}(\alpha)$ which is a unitary group element, not containing the generator $\hat{a}^{\dagger}\hat{a}$ of which $|0\rangle$ is an eigenstate. The substitution of (3.27) in (3.22) leads to, by virtue of the group property, the state of the same form.

The UEVMUCS (3.27) is the so called *Glauber or h.o. coherent state.* On disentangling the exponential in the normal ordered form by employing (2.88), and on using (3.17), (3.27) leads to the expansion

$$|\alpha\rangle = \exp\left(-|\alpha|^2/2\right) \sum_{m=0}^{\infty} \frac{\alpha^m}{\sqrt{m!}}|m\rangle, \tag{3.28}$$

of the coherent state in terms of the number states. The expression (3.10) for the resolution of unity in this case assumes the form

$$\frac{1}{\pi}\int \mathrm{d}^2\alpha|\alpha\rangle\langle\alpha| = I. \tag{3.29}$$

Here the integration extends over the whole complex plane. This result may be verified (i) by using in its left hand side the expansion (3.28), (ii) on carrying the integration using (A.25), and (iii) on invoking the completeness of the number states. A useful expression for the trace of an operator $\hat{A}$ is obtained by operating (3.29) with $\hat{A}$ and carrying the operation of trace. On applying (1.30), we find that

$$\mathrm{Tr}(\hat{A}) = \frac{1}{\pi}\int \mathrm{d}^2\alpha\langle\alpha|\hat{A}|\alpha\rangle. \tag{3.30}$$

Note that, by virtue of the definition (3.24),

$$\langle\alpha|\hat{a}^{\dagger m}\hat{a}^{n}|\alpha\rangle = \alpha^{*m}\alpha^{n}. \tag{3.31}$$

Hence, if it is given that

$$\langle\alpha|\hat{A}|\alpha\rangle = \sum_{m,n} c_{mn}\alpha^{*m}\alpha^{n} \tag{3.32}$$

then the operator form of $\hat{A}$ would read

$$\hat{A} = \sum_{m,n} c_{mn}\hat{a}^{\dagger m}\hat{a}^{n}. \tag{3.33}$$

Now, as a consequence of (3.29), any state $|\psi\rangle$ in the space of the h.o. may be expressed as

$$|\psi\rangle = \frac{1}{\pi}\int \mathrm{d}^2\alpha|\alpha\rangle\langle\alpha|\psi\rangle. \tag{3.34}$$

The complex number $\langle\alpha|\psi\rangle$ represents $|\psi\rangle$ in the basis of the coherent states. On using (3.28) for $|\alpha\rangle$, we see that

$$\langle\alpha|\psi\rangle = \exp\left(-|\alpha|^2/2\right)\sum_{m=0}^{\infty}\frac{\alpha^{*m}}{\sqrt{m!}}\langle m|\psi\rangle \equiv \exp\left(-|\alpha|^2/2\right)\psi(\alpha^*). \tag{3.35}$$

The factor $\exp\left(-|\alpha|^2/2\right)$ is only a normalization factor. The state $|\psi\rangle$ may thus be represented by the function $\psi(\alpha^*)$:

$$\psi(\alpha^*) = (\alpha|\psi\rangle, \qquad |\alpha) = \sum_{m=0}^{\infty}\frac{\alpha^m}{\sqrt{m!}}|m\rangle. \tag{3.36}$$

This implies that

$$\langle m|\psi\rangle = \frac{1}{\sqrt{m!}}\frac{\mathrm{d}^m}{\mathrm{d}z^m}\psi(z)\Big|_{z=0}. \tag{3.37}$$

Now, on taking the matrix element of (3.29) in the states $|\psi\rangle$ and $|\phi\rangle$, the scalar product between the states may be expressed as

$$\langle\phi|\psi\rangle = \frac{1}{\pi}\int \mathrm{d}^2\alpha\langle\phi|\alpha\rangle\langle\alpha|\psi\rangle = \frac{1}{\pi}\int \mathrm{d}^2\alpha\exp\left(-|\alpha|^2\right)\phi(\alpha)\psi(\alpha^*). \tag{3.38}$$

This shows that $|\psi\rangle$ is normalizable if

$$\frac{1}{\pi}\int \mathrm{d}^2\alpha\ \exp\left(-|\alpha|^2\right)|\psi(\alpha)|^2 < \infty. \tag{3.39}$$

By using the definition (3.36) of $|\alpha)$, show that

$$(\alpha|\hat{a}^{\dagger} = \alpha^*(\alpha|, \qquad (\alpha|\hat{a} = \frac{\mathrm{d}}{\mathrm{d}\alpha^*}(\alpha|. \tag{3.40}$$

Accordingly, if $\Phi\left(\hat{a}^{\dagger},\hat{a}\right)$ is a function of the h.o. operators then

$$(\alpha|\Phi\left(\hat{a}^{\dagger},\hat{a}\right)|\psi\rangle = \Phi\left(\alpha^*,\frac{\mathrm{d}}{\mathrm{d}\alpha^*}\right)\psi(\alpha^*). \tag{3.41}$$

Note that this requires the existence of all derivatives of $\psi(\alpha^*)$. That is ensured by the requirement that $\psi(\alpha^*)$ be an entire function. Thus, in the basis

of the coherent states, a state vector is represented by an entire function, normalizable in the sense of (3.39). The action of an operator on a vector is represented as a differential operator on the corresponding function. This is known as the *function-space or Bargmann representation* [42].

We have thus constructed the UEVMUCS by using the solution of (3.23) for $\lambda = 1$. We will see that the UEVMUCS corresponding to $\lambda = -1$ do not exist.

Next, we construct the minimum uncertainty coherent states of a harmonic oscillator corresponding to the solution of (3.23) for $\lambda \neq 1$. That equation, rewritten in terms of $\hat{a}$, $\hat{a}^\dagger$ (by suitably redefining β) reads

$$\left[(1+\lambda)\hat{a} + (1-\lambda)\hat{a}^\dagger\right]|\lambda,\beta\rangle = \sqrt{2}\beta|\lambda,\beta\rangle. \tag{3.42}$$

On applying (3.41), (3.42) assumes the form

$$\left[(1+\lambda)\frac{\mathrm{d}}{\mathrm{d}z} + (1-\lambda)z\right](z^*|\lambda,\beta\rangle = \sqrt{2}\beta(z^*|\lambda,\beta\rangle. \tag{3.43}$$

This is evidently solved by

$$(z^*|\lambda,\beta\rangle = D\exp\left[-\frac{1-\lambda}{2(1+\lambda)}z^2 + \frac{\sqrt{2}\beta}{1+\lambda}z\right] \tag{3.44}$$

with D as the normalization constant. Now, apply (A.29) to show that $(z^*|\lambda,\beta\rangle$ would satisfy the normalizability condition (3.39) if

$$\mathrm{Re}\,(\lambda) > 0. \tag{3.45}$$

This implies in particular that (3.42) does not admit normalizable solution if $\lambda = -1$, i.e. $\hat{a}^\dagger$ does not admit normalizable right eigenvectors. If (3.45) holds then an application of (A.29) shows that

$$\begin{aligned} 1 &= \frac{1}{\pi}\int \mathrm{d}^2\alpha|(z^*|\lambda,\beta\rangle|^2 \\ &= |D|^2\sqrt{\frac{(1+\lambda)(1+\lambda^*)}{2(\lambda+\lambda^*)}}\exp\left(|\beta|^2/(\lambda+\lambda^*)\right) \\ &\quad\times\exp\left[\left(-\beta^2(1-\lambda^*)/\{2(1+\lambda)(\lambda+\lambda^*)\}\right) + \mathrm{c.c.}\right]. \end{aligned} \tag{3.46}$$

This determines the constant D. Now, by using (A.33), (3.44) may be expressed in terms of the Hermite polynomials as

$$(z^*|\lambda,\beta\rangle = D\sum_{m=0}^{\infty}\frac{z^m}{m!}\left(\frac{1-\lambda}{2(1+\lambda)}\right)^{m/2}H_m\left(\frac{\beta}{\sqrt{1-\lambda^2}}\right). \tag{3.47}$$

On combining this with (3.37) it follows that the MUCS satisfying (3.42) is given in the number state representation by

$$|\lambda,\beta\rangle = D\sum_{m=0}^{\infty}\frac{1}{\sqrt{m!}}\left(\frac{1-\lambda}{2(1+\lambda)}\right)^{m/2}H_m\left(\frac{\beta}{\sqrt{1-\lambda^2}}\right)|m\rangle. \tag{3.48}$$

This is normalizable, of course, only if (3.45) holds.

Let us now examine the case $\mathrm{Re}(\lambda) = 0$ which separates the admissible and inadmissible regions. For the sake of illustration, we let $\lambda = \mathrm{i}$ and evaluate $\langle \lambda\beta' | \lambda, \beta\rangle$ using (3.38). Write the integration variable α as $x + \mathrm{i}y$ and carry out the integration over x to obtain

$$\langle \lambda, \beta' | \lambda, \beta\rangle \sim \int_{-\infty}^{\infty} \mathrm{d}y \exp\left[\sqrt{2}\mathrm{i}y(\beta - \beta'^*)\right]. \tag{3.49}$$

Now, if the β's are complex then (3.49) diverges whereas it reduces to a delta function if the β's are real. Hence, the solution of (3.42) is (delta-function) normalizable for $\mathrm{Re}(\lambda) = 0$ if β is real. That is, of course, as it should be because for $\mathrm{Re}(\lambda) = 0$, the eigenvalue equation (3.42) becomes an equation for a hermitian operator whose eigenvalues ought to be real.

An instructive form of the solution of (3.42) is obtained by rewriting it as

$$\hat{a}(\xi)|\xi, \alpha\rangle = \alpha|\xi, \alpha\rangle, \qquad \xi = \exp(\mathrm{i}\phi)|\xi|, \tag{3.50}$$

$$\hat{a}(\xi) = \cosh(|\xi|)\hat{a} + \exp(\mathrm{i}\phi)\sinh(|\xi|)\hat{a}^\dagger. \tag{3.51}$$

On comparing the forms (3.50) and (3.42) we note that

$$\begin{aligned}
\cosh(|\xi|) &= \sqrt{\frac{(1+\lambda)(1+\lambda^*)}{2(\lambda+\lambda^*)}}, \\
\exp(\mathrm{i}\phi)\sinh(|\xi|) &= (1-\lambda)\sqrt{\frac{(1+\lambda^*)}{2(1+\lambda)(\lambda+\lambda^*)}}, \\
\alpha &= \beta\sqrt{\frac{(1+\lambda^*)}{(1+\lambda)(\lambda+\lambda^*)}}.
\end{aligned} \tag{3.52}$$

By employing (2.80), (3.50) may be rewritten as

$$\hat{a}\hat{S}^\dagger(\xi)|\xi, \alpha\rangle = \alpha\hat{S}^\dagger(\xi)|\xi, \alpha\rangle \tag{3.53}$$

where $\hat{S}(\xi)$, called the *squeezing operator*, is defined by

$$\hat{S}(\xi) = \exp\left[\frac{1}{2}\left(\xi^* a^2 - \xi a^{\dagger 2}\right)\right]. \tag{3.54}$$

Comparison of (3.53) and (3.24) implies that

$$|\xi, \alpha\rangle = \hat{S}(\xi)|\alpha\rangle = \hat{S}(\xi)\hat{D}(\alpha)|0\rangle. \tag{3.55}$$

Note that $\hat{S}(\xi)$ is an element, not of the h.o., but of the $SU(1,1)$ group. The meaning of the term squeezing is clarified below. For a detailed discussion of the properties of these states and the quantum optical processes for their generation, see [43, 44].

It is instructive to examine the probability p_m that the state $|\xi, \alpha\rangle$ has excitation number m:

$$p_m = |\langle m|\xi,\alpha\rangle|^2. \tag{3.56}$$

This may be evaluated by using (3.48) and the relations (3.52) between (λ,β) and (ξ,α). In Fig. 3.1 we have plotted p_m as a function of m for $\alpha = 6$, $\sinh(|\xi|) = 3.5$. Notice the oscillatory behaviour of p_m. We compare it with

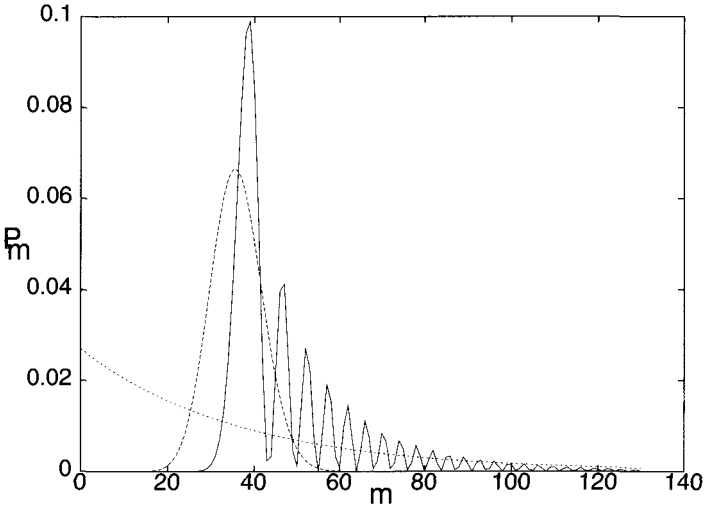

Fig. 3.1. Plot of p_m as a function of m for the oscillator in the squeezed state (3.55) with $\sinh(|\xi|) = 3.5$, $\alpha = 6$ (solid curve). Long dashed curve is for the coherent state with $\alpha = 6$. Short dashed curve is for the thermal state with $\bar{n} = 36$.

$p_m = |\langle m|\alpha\rangle|^2$ in the coherent state and with $p_m = \langle m|\hat{\rho}_{\rm th}|m\rangle$ in the state of the oscillator in equilibrium with a bath at temperature T described by

$$\hat{\rho}_{\rm th} = \exp(-\beta\hat{a}^\dagger\hat{a})/{\rm Tr}[\exp(-\beta\hat{a}^\dagger\hat{a})], \quad \beta = \hbar\omega_0/{\rm k_B}T, \tag{3.57}$$

$\rm k_B$ being the Boltzmann constant and ω_0 the frequency of the oscillator. We also define

$$\exp(-\beta) = \frac{\bar{n}}{\bar{n}+1}. \tag{3.58}$$

Long dashes in the Fig. 3.1 are for the coherent state $|\alpha\rangle$ with $\alpha = 6$ whereas small dashes are for the thermal state corresponding to $\bar{n} = 36$. The p_m for the coherent state is a Poissonian centered at $|\alpha|^2 = 36$. The p_m is a monotonically decreasing function for the thermal state. The oscillatory behaviour of p_m for the squeezed state is a signature of its non-classicality [45] in the sense explained in Chap. 4.

The operator averages in state $|\xi,\alpha\rangle$ may be found by noting that

$$\begin{aligned}\langle\xi,\alpha|F(\hat{a},\hat{a}^\dagger)|\xi,\alpha\rangle &= \langle\alpha|\hat{S}^\dagger(\xi)F(\hat{a},\hat{a}^\dagger)\hat{S}(\xi)|\alpha\rangle \\ &= \langle\alpha|F(\hat{a}(\xi),\hat{a}^\dagger(\xi))|\alpha\rangle,\end{aligned} \tag{3.59}$$

the $\hat{a}(\xi)$ being given by (3.51) with $\xi \to -\xi$. The matrix element in the last line above may be evaluated by expressing F in normal-ordered form in $\hat{a}, \hat{a}^\dagger$. For example, the average occupation number in the state $|\xi, \alpha\rangle$ is

$$\begin{aligned}\langle \hat{a}^\dagger \hat{a} \rangle &= \langle \alpha | \left\{ \cosh(|\xi|)\hat{a}^\dagger - \exp(-\mathrm{i}\phi)\sinh(|\xi|)\hat{a} \right\} \\ &\quad \times \left\{ \cosh(|\xi|)\hat{a} - \exp(\mathrm{i}\phi)\sinh(|\xi|)\hat{a}^\dagger \right\} |\alpha\rangle \\ &= |\alpha|^2 \left\{ \cosh^2(|\xi|) + \sinh^2(|\xi|) \right\} + \sinh^2(|\xi|) \\ &\quad - \cosh(|\xi|)\sinh(|\xi|) \left\{ \exp(-\mathrm{i}\phi)\alpha^2 + \text{c.c.} \right\}. \end{aligned} \tag{3.60}$$

Squeezed States of Harmonic Oscillator. Let

$$\hat{X}(\theta) = \frac{1}{\sqrt{2}} \left[\exp(\mathrm{i}\theta)\hat{a} + \exp(-\mathrm{i}\theta)\hat{a}^\dagger \right]. \tag{3.61}$$

Then $[\hat{X}(\theta + \pi/2),\ \hat{X}(\theta)] = \mathrm{i}$. Using (3.59), verify that

$$\begin{aligned}\Delta X^2(\theta) = \frac{1}{2} \Big[&\cosh^2(|\xi|) + \sinh^2(|\xi|) \\ &-2\cosh(|\xi|)\sinh(|\xi|)\cos(\phi + 2\theta) \Big]. \end{aligned} \tag{3.62}$$

A coherent state corresponds to $\xi = 0$. Hence, in a coherent state, $\Delta X^2(\theta) = 1/2$. A state in which $\Delta X^2(\theta) < 1/2$ for some θ is called a *squeezed state*. We will see in Chap. 4 that a squeezed state is non-classical in the sense explained there. On examining (3.62) we note that it is posible to have $\Delta X^2(\theta) < 1/2$ in the MUS $|\xi, \alpha\rangle$. For example, if $2\theta + \phi = 0$ then $\Delta X^2(\theta) = \exp(-2|\xi|)/2 < 1/2$.

The concept of squeezing introduced above is based on an application of the uncertainty relation. This concept, however, arises in the context of search for minimizing error in a process of measurement imposed by the quantum theory. For, the quantum theory assigns, through the uncertainty relation, an inherent error to the measured value of an observable. This error is independent of the one contributed by external factors like those due to the limitations of the apparatus and the environment. It is, therefore, imperative to know how to minimize the intrinsic quantum imprecision in a process of measurement. By analyzing some simple measurement processes it has been shown that if a process of measurement by means of a h.o. is carried by finding average of an observable $\hat{X}(\theta)$ defined in (3.61) then the fundamental quantum-theoretic error in the measurement of that average is related directly with $\Delta X^2(\theta)$ [43, 46]. Hence, smaller the $\Delta X^2(\theta)$, better is the sensitivity of measurement. Since $\Delta X^2(\theta)$ in a squeezed state is smaller than that in a coherent state, the squeezed state provides better precision in measurement than that in a coherent state. For detailed examples and comparative estimates of errors in various states, see [46].

MUS for the Pair $(\hat{p}, \hat{a}^\dagger\hat{a})$. Next we construct the minimum uncertainty states for the pair $(\hat{p}, \hat{a}^\dagger\hat{a})$ which is equivalent to the pair $(\hat{q}, \hat{a}^\dagger\hat{a})$. The equation (3.12) then reads

$$\left[\hat{a}^\dagger\hat{a} + i\lambda\hat{p}\right]|\psi\rangle = \alpha|\psi\rangle. \tag{3.63}$$

By using (2.31) this may be rewritten as

$$\exp(\lambda\hat{q})\left[\hat{a}^\dagger\hat{a} + \frac{\lambda^2}{2}\right]\exp(-\lambda\hat{q})|\psi\rangle = \alpha|\psi\rangle, \tag{3.64}$$

which is solved by

$$|\psi_m\rangle = \exp(\lambda\hat{q})|m\rangle. \tag{3.65}$$

We leave it to the reader to check that substitution of this for $|\psi\rangle$ in (3.22) results in $|\{\beta\}\rangle$ in the form $\hat{D}(\beta,\lambda)|m\rangle$ where $\hat{D}(\beta,\lambda)$ is an exponent of a linear combination of $\hat{a}, \hat{a}^\dagger$. The operator $\exp(\lambda\hat{q})$ in (3.65), and consequently $\hat{D}(\beta,\lambda)$, are unitary if $\lambda = \pm i$. The state $|\psi_m\rangle$ is then of the form $\hat{D}(\beta)|m\rangle$. These states were investigated in [47] as a generalization of the Glauber coherent states.

We have thus constructed all the unitarily inequivalent MUCS of the harmonic oscillator algebra. Next, we construct such states for the $SU(2)$ algebra.

3.3 Representations of $SU(2)$

Recall from chapter 2 that the $SU(2)$ algebra is generated by the operators $\left(\hat{S}_x, \hat{S}_y, \hat{S}_z\right) \equiv \boldsymbol{S}$ which obey the commutation relations (1.126). It may be verified by inspecting (2.41) that these operators can be transformed to each other by an $SU(2)$ unitary group element. Hence all the generators of this algebra are unitarily equivalent. In this case, therefore, the orthonormal representations corresponding to the eigenstates of any of the three operators are equivalent and so are the continuous bases generated by any pair of generators.

3.3.1 Orthonormal Representation

An $SU(n)$ algebra admits $n-1$ operators, called the Casimir operators, which commute with all the operators of that algebra. For the $SU(2)$ there is one Casimir operator. It is the *total spin operator*

$$\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2. \tag{3.66}$$

The vector space of the algebra is, therefore, reducible to a sum of the subspaces each characterized by an eigenvalue of $\hat{S}^2$. The eigenvalues of $\hat{S}^2$ are known to be given by $S(S+1)$ where $S = 1/2, 1, 3/2, \ldots$ and the eigenvalues of any hermitian $SU(2)$ operator, say $\hat{S}_z$, are given by $m = -S, -S+1, \ldots, S$. Hence, simultaneous eigenstates $|S,m\rangle$ of $\hat{S}^2$ and $\hat{S}_z$ defined by

$$\hat{S}^2|S,m\rangle = S(S+1)|S,m\rangle, \qquad S = 1/2, 1, \ldots$$
$$\hat{S}_z|S,m\rangle = m|S,m\rangle, \qquad m = -S, -S+1, \ldots, S \tag{3.67}$$

constitute a basis for the $2S+1$ dimensional space of the spin states. The set of states $|S,m\rangle$ is complete and orthonormal:

$$\sum_{m=-S}^{S} |S,m\rangle\langle S,m| = I, \quad \langle S,m|S,n\rangle = \delta_{mn}. \tag{3.68}$$

Consider now the operators $\hat{S}_\pm$ defined in (1.127). By expressing $\hat{S}_x$, $\hat{S}_y$ in terms of $\hat{S}_\pm$ and, by using (3.67) to write $\hat{S}^2 = S(S+1)$, (3.66) reads

$$S(S+1) = \frac{1}{2}\left[\hat{S}_+\hat{S}_- + \hat{S}_-\hat{S}_+\right] + \hat{S}_z^2 = \hat{S}_+\hat{S}_- - \hat{S}_z + \hat{S}_z^2. \tag{3.69}$$

Next, using the commutation relations (1.128) of $\hat{S}_\pm$ with $\hat{S}_z$, verify that

$$\hat{S}_+|S,m\rangle = \sqrt{(S-m)(S+m+1)}|S,m+1\rangle,$$
$$\hat{S}_-|S,m\rangle = \sqrt{(S+m)(S-m+1)}|S,m-1\rangle. \tag{3.70}$$

The operators $\hat{S}_\pm$ thus act as raising and lowering operators of the eigenstates of $\hat{S}_z$. By repeated application of (3.70), we find that

$$\hat{S}_+^k|S,m\rangle = \sqrt{\frac{(S-m)!(S+m+k!)}{(S-m-k)!(S+m)!}}|S,m+k\rangle,$$
$$\hat{S}_-^k|S,m\rangle = \sqrt{\frac{(S+m)!(S-m+k)!}{(S+m-k)!(S-m)!}}|S,m-1\rangle. \tag{3.71}$$

The action of $\hat{S}_x, \hat{S}_y$ on $|S,m\rangle$ is determined by expressing them in terms of $\hat{S}_\pm$. Note, in particular, that

$$\hat{S}_+|S,S\rangle = 0, \quad \hat{S}_-|S,-S\rangle = 0, \quad \hat{S}_\pm^{2S+1} = 0. \tag{3.72}$$

As stated before, there is no other unitarily inequivalent set of orthonormal states in this algebra. The eigenstates of any hermitian operator in the algebra can be found by constructing the transformation coupling it with $\hat{S}_z$.

We have already encountered in Chap. 1 the case of a spin-1/2, i.e. of a spin of total spin quantum number $S = 1/2$. We will have occasions to deal also with a collection of N spin-1/2s. Such a collection is described by the *collective spin operators*

$$\hat{S}_\mu = \sum_{i=1}^{N} \hat{S}_\mu^{(i)} \tag{3.73}$$

where $\hat{S}_\mu^{(i)}$ is the μ^{th} component of the spin-1/2 labeled i. It is straightforward to verify that (3.73) obey the commutation relations (1.126). The state space of such a system is evidently spanned by $|m_1, m_2, \ldots, m_N\rangle$,

$m_1, m_2, \ldots, m_N = \pm 1/2$. The set of these states is reducible to a sum of sets of states such that the states in each set transform amongst each other. Each such set is characterized by a total spin quantum number S capable of assuming the values $S = N/2, N/2-1, \ldots, 0$ or $1/2$ depending on whether N is even or odd. The states in a set, characterized by S, may thus be labeled by $|S, m\rangle$ $(m = -S, -S+1, \ldots, S)$.

Next we construct the minimum uncertainty states of $SU(2)$.

3.3.2 Minimum Uncertainty Coherent States

As mentioned above, there is only one pair-class in this case which is generated by any two, say $(\hat{S}_x, \hat{S}_y)$, of the three generators. Its minimum uncertainty coherent states are constructed by substituting for $|\psi\rangle$ in

$$|\{\mu\}\rangle = \exp\left[\mathrm{i}\left\{\mu_x \hat{S}_x + \mu_y \hat{S}_y + \mu_z \hat{S}_z\right\}\right]|\psi\rangle, \tag{3.74}$$

the solution $|\lambda, z\rangle$ of the equation

$$\left[\hat{S}_x + \mathrm{i}\lambda \hat{S}_y\right]|\lambda, z\rangle = z|\lambda, z\rangle \tag{3.75}$$

and by dropping from the exponent in (3.74) the operators of which $|\lambda, z\rangle$ is an eigenstate. The equation (3.75) determines the minimum uncertainty states of the pair $\hat{S}_x, \hat{S}_y$.

Consider first the case of UEVMUS corresponding to $\lambda = \pm 1$. In this case, (3.75) assumes the form

$$\hat{S}_\pm|\pm, z\rangle = z|\pm, z\rangle. \tag{3.76}$$

By virtue of (3.70) and (3.72), it is solvable only if $z = 0$ with $|\pm, 0\rangle = |S, \pm S\rangle$. By substituting this solution in (3.74) and by dropping from the exponent the operator $\hat{S}_z$ of which $|S, \pm S\rangle$ is an eigenstate gives the UEVMUCS of $SU(2)$ called the $SU(2)$ *or spin coherent state*. For the sake of definiteness, we let $|S, S\rangle$ be the fiducial state and rewrite (3.74) as

$$|\theta, \phi\rangle = \hat{U}(\theta, \phi)|S, S\rangle, \tag{3.77}$$

$$\begin{aligned}\hat{U}(\theta, \phi) &= \exp\left[\frac{\theta}{2}\Big(\exp(\mathrm{i}\phi)\hat{S}_- - \exp(-\mathrm{i}\phi)\hat{S}_+\Big)\right] \\ &= \exp\left[\mathrm{i}\theta\Big(\sin(\phi)\hat{S}_x - \cos(\phi)\hat{S}_y\Big)\right],\end{aligned} \tag{3.78}$$

$0 \leq \theta < \pi$, $0 \leq \phi < 2\pi$. Similar results follow if the fiducial state is chosen instead to be the state $|S, -S\rangle$. On disentangling the exponential according to (2.96) and on applying (3.71), (3.77) leads to

$$|\Omega\rangle \equiv |\mu\rangle = \frac{1}{(1+|\mu|^2)^S}\sum_{m=0}^{2S}\mu^m\sqrt{\frac{(2S)!}{(2S-m)!m!}}|S, S-m\rangle, \tag{3.79}$$

where

$$\mu = \exp(\mathrm{i}\phi)\tan\left(\frac{\theta}{2}\right). \tag{3.80}$$

The scalar product between the spin coherent states is given by

$$\langle\nu|\mu\rangle = \frac{1}{(1+|\nu|^2)^S(1+|\mu|^2)^S}\,(1+\nu^*\mu)^{2S}. \tag{3.81}$$

In the following we enumerate some properties of the spin coherent states.

1. The spin coherent state is an eigenstate of the spin component in the direction (θ,ϕ). To see this, operarte the eigenvalue equation (3.67) of $\hat{S}_z$ corresponding to $m = S$ by $\hat{U}$ and rewtrite it as

$$\left[\hat{U}(\theta,\phi)\hat{S}_z\hat{U}^\dagger(\theta,\phi)\right]\hat{U}(\theta,\phi)|S,S\rangle = S\hat{U}(\theta,\phi)|S,S\rangle. \tag{3.82}$$

Evaluate the similarity transformation above by using (2.42) and invoke the definition (3.77) to get

$$\hat{S}(\theta,\phi)|\theta,\phi\rangle = S|\theta,\phi\rangle, \tag{3.83}$$

$$\begin{aligned}\hat{S}(\theta,\phi) &= \hat{U}(\theta,\phi)\hat{S}_z\hat{U}^\dagger(\theta,\phi)\\ &= \sin(\theta)\cos(\phi)\hat{S}_x + \sin(\theta)\sin(\phi)\hat{S}_y + \cos(\theta)\hat{S}_z.\end{aligned} \tag{3.84}$$

This is, of course, the component of the spin in the direction (θ,ϕ). The equation (3.83) shows that the spin coherent state is an eigenstate corresponding to the eigenvalue S of the spin component in the direction (θ,ϕ).

2. Expectation values of operators in a spin coherent state may be evaluated by noting that

$$\begin{aligned}\langle\theta,\phi|F\left(\{\hat{S}_\mu\}\right)|\theta,\phi\rangle &= \langle S,S|\hat{U}^\dagger(\theta,\phi)F\left(\{\hat{S}_\mu)\}\right)\hat{U}(\theta,\phi)|S,S\rangle\\ &= \langle S,S|F\left(\{\hat{S}_\mu\}(-\theta,\phi)\right)|S,S\rangle,\end{aligned} \tag{3.85}$$

$$\hat{S}_\mu(\theta,\phi) = \hat{U}(\theta,\phi)\hat{S}_\mu\hat{U}^\dagger(\theta,\phi). \tag{3.86}$$

The similarity transformation above may be carried using the results of Sect. 2.3.2.

3. Consider a spin component $\hat{S}^\perp = e_x\hat{S}_x + e_y\hat{S}_y$, $(e_x^2+e_y^2=1)$, in a direction orthogonal to the z direction. Note that

$$\langle\theta,\phi|\hat{S}^\perp(\theta,\phi)|\theta,\phi\rangle = \langle S,S|\hat{S}^\perp|S,S\rangle = 0. \tag{3.87}$$

By virtue of the fact that $\hat{S}^\perp(\theta,\phi)$ is obtained as a result of a unitary transformation of the spin component perpandicular to $\hat{S}_z$, it follows that $\hat{S}^\perp(\theta,\phi)$ is perpandicular to $\hat{S}(\theta,\phi)$, i.e. in a direction orthogonal to the direction of the spin coherent state. The expression (3.87) shows that the average of a spin component in a direction orthogonal to the direction

of a spin coherent state is zero in that state. We evaluate fluctuations in the said orthogonal components by starting with the equation

$$\begin{aligned}&\langle S,S|(e_x\hat{S}_x+e_y\hat{S}_y)^2|S,S\rangle\\&=\langle S,S|\hat{S}_+\hat{S}_-+\hat{S}_-\hat{S}_+|S,S\rangle/4=S/2.\end{aligned}\tag{3.88}$$

In writing this we have made use of (3.69). This implies that

$$\begin{aligned}\langle S,S|\hat{U}^\dagger(\theta,\phi)\hat{S}^{\perp 2}(\theta,\phi)\hat{U}(\theta,\phi)|S,S\rangle&=\langle\theta,\phi|\hat{S}^{\perp 2}(\theta,\phi)|\theta,\phi\rangle\\&=\frac{S}{2}.\end{aligned}\tag{3.89}$$

This, together with (3.87) shows that the variance $\Delta S^{\perp 2}$ in any component orthogonal to the direction of the spin coherent state has the same value $S/2$.

4. The expression for the resolution of identity in the present case reads

$$\frac{2S+1}{\pi}\int\frac{\mathrm{d}^2\mu}{(1+|\mu|^2)^2}\,|\mu\rangle\langle\mu|=\frac{2S+1}{4\pi}\int\mathrm{d}\Omega\,|\Omega\rangle\Omega|=I,\tag{3.90}$$

$\mathrm{d}\Omega=\sin(\theta)\mathrm{d}\theta\mathrm{d}\phi$. Using this relation, a state $|\psi\rangle$ is represented by $\langle\mu|\psi\rangle$ or by the function $\psi(\mu^*)=(\mu|\psi\rangle$ of μ^* where $|\mu)$ is unnormalized spin coherent state:

$$\psi(\mu^*)=(\mu|\psi\rangle,\quad|\mu)=\sum_{m=0}^{2S}\mu^m\sqrt{\frac{(2S)!}{(2S-m)!m!}}|S,S-m\rangle.\tag{3.91}$$

On invoking (3.67) and (3.70), it may also be verified that

$$\begin{aligned}&(\mu|\hat{S}_+=\frac{\mathrm{d}}{\mathrm{d}\mu^*}(\mu|,\quad(\mu|\hat{S}_-=\left[2S\mu^*-\mu^{*2}\frac{\mathrm{d}}{\mathrm{d}\mu^*}\right](\mu|,\\&(\mu|\hat{S}_z=\left(S-\mu^*\frac{\mathrm{d}}{\mathrm{d}\mu^*}\right)(\mu|.\end{aligned}\tag{3.92}$$

Thus, in the spin coherent states representation, vectors are represented by a complex valued function and operators on a vector by the differential operators. Since $S_\pm^{2S+1}=0$, the expressions (3.92) imply that $\psi(\mu^*)$ is a polynomial of degree $2S$ in μ^*.

We have thus constructed the UEVMUCS corresponding to the solution of (3.75) for $\lambda=\pm1$. Its solution for $\lambda\neq\pm1$ my be derived by writing it as

$$\left[\exp(-\theta)\hat{S}_-+\exp(\theta)\hat{S}_+\right]|\theta,z\rangle=z|\theta,z\rangle.\tag{3.93}$$

Use (2.44) and (2.42) to reexpress this as

$$\hat{S}_z\hat{T}|\theta,z\rangle=\frac{z}{2}\hat{T}|\theta,z\rangle,\tag{3.94}$$

where

$$\hat{T}=\exp\left(\mathrm{i}\frac{\pi}{2}\hat{S}_y\right)\exp\left(-\theta\hat{S}_z\right).\tag{3.95}$$

The equation (3.94) shows that $\hat{T}|\theta, z\rangle$ is an eigenstate of $\hat{S}_z$. Hence $z = 2n$ $(-S \leq n \leq S)$ and, with $|\theta, z\rangle \to |\theta, n\rangle$,

$$|\theta, n\rangle = \hat{T}^{-1}|S, n\rangle = \exp(\theta S_z) \exp\left(-\mathrm{i}\frac{\pi}{2}\hat{S}_y\right)|S, n\rangle. \tag{3.96}$$

This is a *squeezed spin state.* The meaning of squeezing of a spin state is discussed below. This state will turn out to be non-classical in the sense explained in Chap. 4. The properties of these states are studied in details in [48].

Next, we derive the expression for $|\theta, n\rangle$ in terms of the eigenststes of $\hat{S}_z$. To that end, let

$$|\theta, n\rangle = \sum_{m=-S}^{S} C_{nm}|S, m\rangle = \sum_{p=0}^{2S} C_{nS-p}|S, S-p\rangle \tag{3.97}$$

The C_{nm} may be determined by substituting this in (3.93) to construct the recursion relation for C_{nm}. It is straightforward to verify that $(N = 2S)$

$$\begin{aligned}&\exp(-\theta)\sqrt{p(N-p+1)}C_{nS-p+1} + \exp(\theta)\sqrt{(p+1)(N-p)}C_{nS-p-1}\\ &= 2nC_{nS-p}.\end{aligned} \tag{3.98}$$

Define

$$f_{np} = C_{nS-p}/\sqrt{p!(N-p)!}. \tag{3.99}$$

On substituting this in (3.98) we obtain

$$\exp(-\theta)(N-p+1)f_{np-1} + \exp(\theta)(p+1)f_{np+1} = 2nf_{np}. \tag{3.100}$$

This is the same as the recursion relation (10.66) solved in Chap. 10. The exact solution (10.69) of (10.66) in the present case assumes the form

$$\begin{aligned}f_{np} &= (-)^p A_n \exp(-p\theta)\\ &\quad \times \sum_k \left[\frac{(-)^k}{(p-k)!k!(n+N/2-k)!(N/2-n-p+k)!}\right].\end{aligned} \tag{3.101}$$

The constant A_n is to be found by the normalization condition. We leave it to the reader to derive this result directly by using (3.96). Note that $p_m = |C_{nm}|^2$ gives the probability of finding the spin in the state $|m\rangle$. We substitute (3.101) in (3.99) and evaluate p_m. The numerical results for p_m as a function of m for a system of $N = 20$ spins in the state $|\theta, 0\rangle$ with $\exp(2\theta) = 5$ are presented by a solid line in Fig. 3.2. We notice that, much like p_m as a function of m in Fig. 3.1 for a harmonic oscillator in its squeezed state, p_m in Fig. 3.2 for the squeezed spin state exhibits an oscillatory behaviour. We will see in Chap. 12 that $|\theta, 0\rangle$ is the steady state of a system of even number of two-level atoms in contact with a squeezed reservoir. We compare this with the equilibrium state of N spins in contact with a thermal reservoir. Such a spin state is characterized by

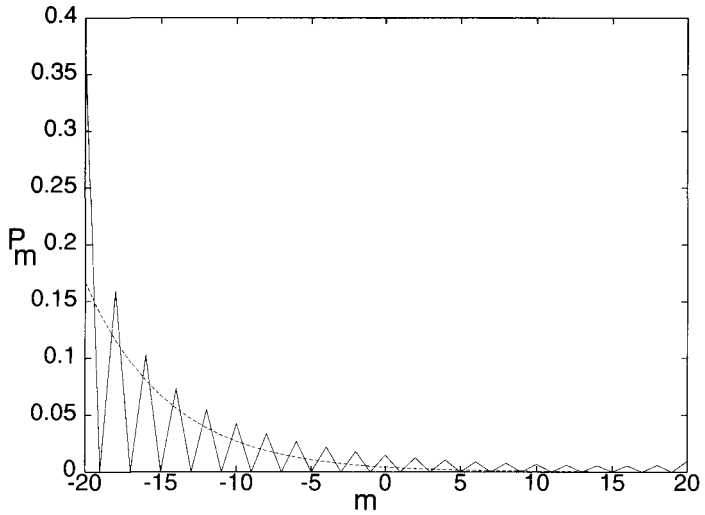

Fig. 3.2. Plot of p_m as a function of m for the spins in the squeezed state (3.96) with $\exp(2\theta) = 5$, $n = 0$, $N = 20$ (solid curve). The dashed curve is for the thermal state with $\bar{n} = 5$.

$$\hat{\rho}_{\rm th} = \exp(-\beta \hat{S}_z)/{\rm Tr}[\exp(-\beta \hat{S}_z)], \tag{3.102}$$

the β being as in (3.57). The corresponding probability of occupation of the state $|S, m\rangle$ is

$$p_m = \langle m|\hat{\rho}_{\rm th}|m\rangle = \exp(-\beta m)/{\rm Tr}[\exp(-\beta \hat{S}_z)], \tag{3.103}$$

$$\begin{aligned} {\rm Tr}(\exp(-\beta \hat{S}_z) &= \exp(-\beta S) \sum_{p=0}^{N} \exp(\beta p) \\ &= \exp(-\beta S) \frac{\exp\{(N+1)\beta\} - 1}{\exp(\beta) - 1}. \end{aligned} \tag{3.104}$$

The behaviour of p_m as a function of m given by (3.103) is displayed by a dashed line in Fig. 3.2 for $\bar{n} = 5$ ($\bar{n}$ defined as in (3.58)) and $N = 20$. The oscillatory behaviour of p_m is a signature of non-classicality of the state in question.

Spin Squeezing. Recall from the discussion of Sect. 3.2 that a squeezed state of the h.o. is defined as the state in which the variance in a linear combination of $\hat{q}$ and $\hat{p}$ is less than its value in the coherent state. Extension of this definition to a spin, however, requires more careful considerations because, whereas the variance of a linear combination of $\hat{q}$ and $\hat{p}$ in a h.o. coherent $|\alpha\rangle$ state is independent of α, the variance in the generators of spin in a spin coherent state $|\theta, \phi\rangle$ depends on (θ, ϕ).

In order to extend the concept of squeezing to a system of spins, we note from (3.89) that the variance in a spin component in any direction orthogonal

to the average direction (θ, ϕ) of a spin in its coherent state $|\theta, \phi\rangle$ is $S/2$. A squeezed spin state may then be defined as a state in which the variance $\Delta S_\mu^{\perp 2}$ in a spin component in some direction orthogonal to the average direction $\langle \hat{\boldsymbol{S}} \rangle$ in that state is less than $|\langle \hat{\boldsymbol{S}} \rangle|/2$ [49]:

$$(\Delta S_\mu^\perp)^2 < \frac{|\langle \hat{\boldsymbol{S}} \rangle|}{2} \qquad \text{(SQ(I))}. \tag{3.105}$$

The concept of spin squeezing may also be based on the problem of sensitivity of measurement. Some simple processes of measurement using a spin have been analyzed [39, 50, 51]. For an experiment on measuring the quantum noise in a spin system, see [52]. In those processes, the measurement of a quantity α is carried by coupling it with linear spin operators. The value of α is related with the averages of an observable $\hat{S}_a$ associated with the component of spin in direction $\boldsymbol{a}$. It is then shown that the error in the measurement of α is given by

$$\Delta\alpha = \Delta S_a / |\langle \hat{S}_a^\perp \rangle| \tag{3.106}$$

where $\hat{S}_a^\perp$ is the observable corresponding to the spin component in a direction orthogonal to $\boldsymbol{a}$. The actual direction depends on the details of the interaction. It can be verified that the minimum value of $(\Delta\alpha)^2$ in the spin coherent state is $1/2S$. Hence the parameter

$$\xi = 2S(\Delta S_a)^2 / |\langle \hat{S}_a^\perp \rangle|^2 \tag{3.107}$$

is defined as a measure of sensitivity of measurements involving a spin. As remarked before, its minimum value in a spin coherent state is $\xi_{\text{min,coh}} = 1$. A squeezed spin state may be defined as the one in which ξ is less than unity, i.e. a state in which

$$\xi = 2S(\Delta S_a)^2 \Big/ \left|\langle \hat{S}_a^\perp \rangle\right|^2 < 1 \quad \text{(SQ(II))}. \tag{3.108}$$

This criterion of squeezing is not the same as the criterion SQ(I) of (3.105). However, since the average of any spin component is less than S, it follows that SQ(II) implies SQ(I). The criterion SQ(II) is also referred to as *spectroscopic squeezing*. The squeezed spin state (3.96) satisfies SQ(I) as well as SQ(II) [53].

Note that $\hat{U}$ in the definition (3.77) of the spin coherent state is expressible as a product of N spin operators. The state $|S, S\rangle$ therein is also a product of N spin-1/2 states. Hence the spins in the state $|\theta, \phi\rangle$ are uncorrelated. However, although the operator $\hat{T}$ defined in (3.95) is a product of individual spins but not the state $|s, n\rangle$ if $n \neq \pm s$. Hence, spins in MUS (3.96) are correlated if $n \neq \pm s$. In general, the spins in states which satisfy SQ(I) or SQ(II) are correlated. For details of evaluation of ξ and the spin-spin correlation function for various states, see [53].

3.4 Representations of $SU(1,1)$

Recall from Chap. 2 that the $SU(1,1)$ algebra is generated by $\hat{\boldsymbol{K}} \equiv (\hat{K}_x, \hat{K}_y, \hat{K}_z)$ which obey the commutation relations (2.47). Note from Sect. 2.3.3 that the norm of vectors in a direction which is a linear combination of x and y directions is positive but the one in the direction z is negative. Hence, linear combinations of directions x and y are equivalent to each other but not to z. Accordingly, the compnents $\hat{K}_x, \hat{K}_y$ are equivalent to each other but not with $\hat{K}_z$. A third inequivalent class is formed by combination of $\hat{K}_z$ with $\hat{K}_x$ or $\hat{K}_y$ which gives zero norm. There are thus three classes of orthogonal bases generated by the hermitian operators of the algebra: one equivalent to the eigenbasis of $\hat{K}_z$, second equivalent to that of $\hat{K}_x$ and third equivalent to $\hat{K}_x + \hat{K}_z$. The Casimir invariant for $SU(1,1)$ is

$$\hat{Q} = \hat{K}_x^2 + \hat{K}_y^2 - \hat{K}_z^2 = \hat{K}_+\hat{K}_- - \hat{K}_z^2 + \hat{K}_z. \tag{3.109}$$

The vector space of $SU(1,1)$ is, therefore, reducible to a sum of invariant subspaces each labeled by an eigenvalue of $\hat{Q}$. Here we are concerned only with one-mode and two-mode bosonic realizations of $SU(1,1)$ defined in (2.71) and (2.76). Verify that the Casimir operator (3.109) in the two-mode bosonic representation assumes the form

$$\hat{Q} = \hat{K}(1-\hat{K}), \quad \hat{K} = \frac{1}{2}(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b} + 1), \tag{3.110}$$

whereas for one-mode realization it reads

$$\hat{Q} = \frac{3}{16}. \tag{3.111}$$

The operator $\hat{K}$ in (3.110) is related with the difference in the occupation number of the two modes which, as a consequence of the fact that $\hat{K}$ is an invariant, remains unchanged in an $SU(1,1)$ process.

3.4.1 Orthonormal Bases

As stated before, in this case there are three inequivalent representations: one equivalent to the eigenbasis of $\hat{K}_z$, second to that of $\hat{K}_x$, and third to $\hat{K}_x + \hat{K}_z$.

Consider the eigenvectors of $\hat{K}_z$. The expressions (2.71) and (2.76) for $\hat{K}_z$ in the bosonic representations and the corresponding expressions (3.110) and (3.111) for $\hat{Q}$ suggest that the eigenstates $|\chi(m,K)\rangle$ of $\hat{K}_z$ are such that

$$\begin{aligned} \hat{Q}|\chi(m,K)\rangle &= K(1-K)|\chi(m,K)\rangle, \\ \hat{K}_z|\chi(m,K)\rangle &= (m+K)|\chi(m,K)\rangle, \quad m = 0,1,2,\ldots \end{aligned} \tag{3.112}$$

where, for the two-mode realization,

$$|\chi(m,K)\rangle \equiv |m+2K-1,m\rangle, \tag{3.113}$$

the $|m,n\rangle$ being a simultaneous eigenstate of the operators $\hat{a}^\dagger\hat{a}$ and $\hat{b}^\dagger\hat{b}$ with m and n as respective eigenvalues. For one-mode realization,

$$|\chi(m,K)\rangle = |2m\rangle, |2m+1\rangle. \tag{3.114}$$

Also, by comparing (3.112) with (3.111), check that $K = 1/4, 3/4$ for one-mode realization. It may also be verified by invoking the definitions (2.71) and (2.76) that

$$\begin{aligned}\hat{K}_-|\chi(m,K)\rangle &= \sqrt{m(m+2K-1)}|\chi(m-1,K)\rangle,\\ \hat{K}_+|\chi(m,K)\rangle &= \sqrt{(m+1)(m+2K)}|\chi(m+1,K)\rangle.\end{aligned} \tag{3.115}$$

Note that

$$\hat{K}_-|\chi(0,K)\rangle = 0. \tag{3.116}$$

Repeated application of (3.115) yields

$$\begin{aligned}\hat{K}_-^n|\chi(m,K)\rangle &= \sqrt{\frac{m!\Gamma(m+2K)}{(m-n)!\Gamma(m+2K-n)}}|\chi(m-n,K)\rangle,\\ \hat{K}_+^n|\chi(m,K)\rangle &= \sqrt{\frac{(m+n)!\Gamma(m+2K+n)}{m!\Gamma(m+2K)}}|\chi(m+n,K)\rangle.\end{aligned} \tag{3.117}$$

Note that, in the single-mode case, $K_\pm$ couple $|m\rangle$ with $|m\pm 2\rangle$. Hence the space of states in this case reduces to a sum of odd and even number states. Also, from the relationship (2.76) between $\hat{K}_z$ and $\hat{a}^\dagger\hat{a}$, we see that $\hat{K}_z|2m\rangle = (m+1/4)|2m\rangle$, $\hat{K}_z|2m+1\rangle = (m+3/4)|2m+1\rangle$. On comparing these results with (3.112) we infer that $K = 1/4$ ($K = 3/4$) for even (odd) number states leading to the correspondence

$$|\chi(m,1/4)\rangle \to |2m\rangle, \quad |\chi(m,3/4)\rangle \to |2m+1\rangle. \tag{3.118}$$

The eigenstates of $\hat{K}_x$ in terms of the eigenstates of $\hat{K}_z$ introduced above are obtained as the special case $\lambda = 0$ of (3.119) below.

3.4.2 Minimum Uncertainty Coherent States

The classes of minimum uncertainty states of $SU(1,1)$ are: One corresponding to the pair $(\hat{K}_x, \hat{K}_y)$ and another to the pair class $(\hat{K}_x, \hat{K}_z)$.

MUS of the Pair $(\hat{K}_x, \hat{K}_y)$. The MUS in this case are the solutions of

$$\left[\hat{K}_x - \mathrm{i}\lambda\hat{K}_y\right]|\lambda,z) = z|\lambda,z\rangle. \tag{3.119}$$

Consider first the case of the UEVMUS corresponding to $\lambda = 1$. The equation (3.119) in this case reduces to the eigenvalue equation (with $|1,z\rangle \equiv |z\rangle$)

$$\hat{K}_-|z\rangle = z|z\rangle \tag{3.120}$$

for the lowering operator. Let

$$|z\rangle = \sum_{m=0}^{\infty} C_m |\chi(m,K)\rangle. \tag{3.121}$$

Substitute this in (3.120) and use (3.115) to arrive at the recursion relation

$$\sqrt{(m+1)(m+2K)}C_{m+1} = zC_m. \tag{3.122}$$

This is easily solvable. Its solution, on substitution in (3.121) yields

$$|z\rangle = A \sum_{m=0} \frac{z^m}{\sqrt{\Gamma(m+1)\Gamma(m+2K)}} |\chi(m,K)\rangle, \tag{3.123}$$

A being the normalization constant. The state

$$|\xi,\eta,z\rangle = \exp\left[\xi\hat{K}_+ - \xi^*\hat{K}_- + \mathrm{i}\eta\hat{K}_z\right]|z\rangle, \ \eta \text{ real} \tag{3.124}$$

generated by the $SU(1,1)$ group transformation is, by definition, the $SU(1,1)$ UEVMUCS. A special case of (3.124) of particular interest is the state $|\xi,\eta,0\rangle$ corresponding to $|z=0\rangle = |\chi(0,K)\rangle$. The $|\chi(0,K)\rangle$ is an eigenstate of $\hat{K}_z$. The relevant part of the coherent state (3.124) then is

$$|\xi\rangle = \exp\left[\xi\hat{K}_+ - \xi^*\hat{K}_-\right]|\chi(0,K)\rangle \tag{3.125}$$

Note that for single-mode realization of $SU(1,1)$ this is the same as the MUS $|\xi,0\rangle$ of the harmonic osciilator defined in (3.55). Now, disentangle the exponential above using (2.104) and expand the resulting exponentials. Next, use (3.116), (3.112) and (3.117) to obtain (with $\mu = \exp(\mathrm{i}\theta)\tanh(|\xi|)$)

$$|\mu\rangle = \left(1-|\mu|^2\right)^K \sum_{m=0}^{\infty} \mu^m \sqrt{\frac{\Gamma(2K+m)}{\Gamma(m+1)\Gamma(2K)}} |\chi(m,K)\rangle. \tag{3.126}$$

The expression (3.10) for the resolution of unity in this case assumes the form

$$\frac{1}{\Gamma(2K-1)} \int \frac{\mathrm{d}^2\mu}{\left(1-|\mu|^2\right)^2} |\mu\rangle\langle\mu| = I. \tag{3.127}$$

Here the integration is over the unit disc $|\mu|^2 < 1$. This relation may be verified by expanding the states as in (3.126) and on carrying the integration.

The resolution of unity is also admitted by the states $|z\rangle$ of (3.123) [54]. The states $|z\rangle$ thus fulfill minimum requirements listed in Sect. 3.1 for labeling a state as a coherent state. Note that this is an eigenstate of the annihilation operator of $SU(1,1)$ like the Glauber coherent state is an eigenstate of the annihilation operator of the h.o. The states $|z\rangle$ of (3.123) have, therefore, been recognized as the $SU(1,1)$ coherent states parameterized by z [54]. The properties of their two-mode realization and quantum optical processes for their generation have been identified by Agarwal [55] who named them *pair coherent states*. Note, however, that the states $|z\rangle$ are not coherent in the sense of Perelomov because the variable z is not a group parameter. That is in contrast with the fact that the parameter α defining the h.o. coherent state

$|\alpha\rangle$ is the h.o. group parameter. Recall also that the action of a h.o. group transformation on a h.o. coherent state $|\alpha\rangle$ results in another h.o. coherent state. However, the state $|z\rangle$ of (3.123) does not transform to another $|z'\rangle$ under the action of an $SU(1,1)$ transformation. The state $|z\rangle$ is, therefore, not a group parameter related coherent state.

We reserve the name $SU(1,1)$ coherent states for the UEVMUCS (3.125).

The completeness relation (3.127) shows that any $SU(1,1)$ state $|\psi\rangle$ may be represented by $\langle\mu|\psi\rangle$ or by the function $\psi(\mu^*) = (\mu|\psi\rangle$ of μ^* where $|\mu)$ is the unnormalized $SU(1,1)$ coherent state:

$$\psi(\mu^*) = (\mu|\psi\rangle, \quad |\mu) = \sum_{m=0}^{\infty} \mu^m \sqrt{\frac{\Gamma(2K+m)}{\Gamma(m+1)\Gamma(2K)}} |\chi(m,K)\rangle. \tag{3.128}$$

On using (3.127), the scalar product of two states in this representation turns out to be given by

$$\langle\phi|\psi\rangle = \frac{1}{\Gamma(2K-1)} \int \mathrm{d}^2\mu \left(1-|\mu|^2\right)^{2K-2} \phi(\mu)\, \psi(\mu^*). \tag{3.129}$$

This shows that $|\psi\rangle$ is normalizable if

$$\frac{1}{\Gamma(2K-1)} \int \mathrm{d}^2\mu \left(1-|\mu|^2\right)^{2K-2} |\psi(\mu)|^2 < \infty. \tag{3.130}$$

On using (3.115) and (3.128), it is easy to show that

$$\begin{aligned} (\mu|\hat{K}_- &= \frac{\partial}{\partial\mu^*}(\mu|, \quad (\mu|\hat{K}_+ = \left[\mu^{*2}\frac{\partial}{\partial\mu^*} + 2K\mu^*\right](\mu| \\ (\mu|\hat{K}_z &= \left[\mu^*\frac{\partial}{\partial\mu^*} + K\right](\mu|. \end{aligned} \tag{3.131}$$

These relations are useful in converting the $SU(1,1)$ operator equations in to differential equations by means of the correspondence

$$\begin{aligned} &(\mu|\Phi\left(\hat{K}_+, \hat{K}_-, \hat{K}_z\right)|\psi\rangle \\ &= \Phi\left(\frac{\partial}{\partial\mu^*}, \mu^{*2}\frac{\partial}{\partial\mu^*} + 2K\mu^*, \mu\frac{\partial}{\partial\mu^*} + K\right)\psi(\mu^*). \end{aligned} \tag{3.132}$$

This, along with (3.130), demands $\psi(\mu)$ to be analytic in the disc $|\mu|^2 < 1$ if it is to be an admissible function.

The representation (3.132) along with the analyticity requirement provides a simple means for solving $SU(1,1)$ equations. This method has been used in [40, 41] to solve (3.119) for $\lambda \neq 1$ and to find the MUS of the other pair class, namely, the pair $(\hat{K}_z, \hat{K}_x)$. We conclude by mentioning that (3.119) admits normalizable solutions only if $\mathrm{Re}(\lambda) > 0$ and that its solution is delta-function normalizable if $\mathrm{Re}(\lambda) = 0$. Hence (3.119) does not admit acceptable solution for $\lambda = -1$. Consequently, the states (3.125) corresponding to $\lambda = 1$ are the only UEVMUCS of $SU(1,1)$ for the pair $(\hat{K}_x, \hat{K}_y)$. We refer to [40, 41] for details.

4. Quasiprobabilities and Non-classical States

Recall that a classical dynamical system may be described by a phase space probability distribution function $f(\{q\},\{p\})$, ($\{q\} \equiv q_1, q_2, \cdots, q_N$; $\{p\} \equiv p_1, p_2, \cdots, p_N$) which is such that $f(\{q\},\{p\})d^N q d^N p$ gives the probability that the system is in a volume element $d^N q d^N p$ centered around $(\{q\},\{p\})$. In the quantum mechanical description of a dynamical system, however, the phase space coordinates q_i and p_i can not be ascribed definite values simultaneously . Hence the concept of phase space distribution function does not exist for a quantum system. It is, however, possible to construct for a quantum system functions, called quasiprobability distributions (QPDs), resembling the classical phase space distribution functions. A QPD provides insight into quantum-classical correspondence as well as useful means of calculations.

4.1 Phase Space Distribution Functions

For the sake of simplicity, consider a one dimensional dynamical system described classically by a phase space distribution function $f(q,p,t)$. It determines the average of any function $A(q,p)$ by means of the relation

$$\langle A(q,p)\rangle_{\rm cl} = \int {\rm d}q{\rm d}p A(q,p) f(q,p,t). \tag{4.1}$$

The quantum mechanical description of a system, on the other hand, is contained in its density operator $\hat{\rho}$ which determines the average of any function $\hat{A}(\hat{q},\hat{p})$ by the relation

$$\langle \hat{A}(\hat{q},\hat{p})\rangle_{\rm qm} = {\rm Tr}\Big[\hat{\rho}\hat{A}(\hat{q},\hat{p})\Big]. \tag{4.2}$$

Now, assume that $\hat{A}$ is a function of $\hat{q}$ alone and carry the operation of trace in the equation above in the basis of the eigenstates $|q\rangle$ of $\hat{q}$ to get

$$\langle \hat{A}(\hat{q})\rangle_{\rm qm} = \int {\rm d}q A(q)\langle q|\hat{\rho}|q\rangle. \tag{4.3}$$

This is of the form (4.1) of the classical phase space average. Similar result follows by working in the basis of the eigenstates $|p\rangle$ of $\hat{p}$ in case $\hat{A}$ is a function only of $\hat{p}$. However, as argued in Sect. 1.1, there is no state which

is simultaneously an eigenstate of $\hat{q}$ and $\hat{p}$. Hence the foregoing procedure can not yield an analog of the classical phase space distribution function in quantum mechanics.

However, through (4.2), the density operator determines the average of any function of the operators $\hat{q}$, $\hat{p}$. Note also that the classical distribution function may be expressed in terms of the averages of a complete set of functions of q and p. This suggests that we may be able to construct a quantum analog of the classical distribution function by expressing the latter in terms of the average of a suitably chosen complete set of functions and by identifying those classical averages as quantum mechanical ones. In order to explore this possibility we rewrite a classical distribution as

$$\begin{aligned} f(q,p,t) &= \int \mathrm{d}q'\mathrm{d}p'\delta(q-q')\delta(p-p')f(q',p',t) \\ &= \frac{1}{4\pi^2}\int \mathrm{d}q'\mathrm{d}p'\mathrm{d}k\mathrm{d}l\exp[\mathrm{i}\{k(q-q')+l(p-p')\}]f(q',p',t) \\ &= \frac{1}{4\pi^2}\int \mathrm{d}k\mathrm{d}l\exp(\mathrm{i}kq)\exp(\mathrm{i}lp)\langle\exp(-\mathrm{i}kq)\exp(-\mathrm{i}lp)\rangle_{\mathrm{cl}}. \end{aligned} \tag{4.4}$$

This expresses the distribution function $f(q,p,t)$ in terms of the average of a complete set of functions of q and p. Now, to construct the quantum analog of $f(q,p,t)$, (i) express the exponential under the average in (4.4) as a sum of products of the form $q^m p^n$, (ii) replace the c-numbers q,p by the operators $\hat{q},\hat{p}$, and (iii) replace the classical average by the quantum average defined as in (4.2). The basic difficulty in administering this prescription lies, however, in the fact that, due to non-commutivity of $\hat{q},\hat{p}$, there are several different operator forms of a c-number product $q^m p^n$ if $m,n \neq 0$. Those different forms correspond to different ways of ordering $\hat{q}$ and $\hat{p}$. For example, q^2p may be represented by any of the forms: $\hat{q}^2\hat{p}$, $\hat{q}\hat{p}\hat{q}$, $\hat{p}\hat{q}^2$ or by their linear combination $x_1\hat{q}^2\hat{p}+x_2\hat{q}\hat{p}\hat{q}+x_3\hat{p}\hat{q}^2$ where x_i are arbitrary subject to the condition $x_1+x_2+x_3=1$. This condition ensures that the linear combination in question reduces to q^2p when the operators are replaced by c-numbers. In general, we formally represent a c-number product as an operator as

$$q^m p^n \to \Omega\,(\hat{q}^m\hat{p}^n) \tag{4.5}$$

where $\Omega(\hat{q}^m\hat{p}^n)$ defines a linear combination of m $\hat{q}$'s and n $\hat{p}$'s. Use the correspondence (4.5) to replace the exponential of c-numbers under the average in (4.4) and follow the step (iii) above to get

$$\begin{aligned} f^{(\Omega)}(q,p,t) = \frac{1}{4\pi^2}\int \mathrm{d}k\mathrm{d}l\exp(\mathrm{i}kq)\exp(\mathrm{i}lp) \\ \times\left\langle\Omega\Big[\exp(-\mathrm{i}k\hat{q})\exp(-\mathrm{i}l\hat{p})\Big]\right\rangle_{\mathrm{qm}}. \end{aligned} \tag{4.6}$$

This is a quantum analog of the classical phase space distribution function. Different choices of the correspondence Ω lead to different $f^{(\Omega)}(q,p,t)$, each

called a *quasiprobability distribution* (QPD). It is designated a quasiprobability to emphasize the fact that it is a mathematical construct and not a true phase space distribution function as no such function exists for a quantum system. Let us now examine the results of different ordering prescriptions.

In order to investigate various operator orderings, it is convenient to express $\hat{q}, \hat{p}$ in terms of the creation and annihilation operators $\hat{a}, \hat{a}^\dagger$ and to transform suitably to complex variables so as to rewrite (4.6) as

$$f^{(\Omega)}(\alpha, \alpha^*) = \frac{1}{\pi^2} \int \mathrm{d}^2\xi \exp\left[\mathrm{i}(\alpha\xi + \alpha^*\xi^*)\right] \times \mathrm{Tr}\left[\Omega\left\{\exp(-\mathrm{i}\hat{a}\xi)\exp(-\mathrm{i}\hat{a}^\dagger\xi^*)\right\}\hat{\rho}\right]. \tag{4.7}$$

Now, let

$$\Omega\left\{\exp(\mathrm{i}\xi\hat{a})\exp(\mathrm{i}\xi^*\hat{a}^\dagger)\right\} = \Pi_{j=1}^{N}[\exp(\mathrm{i}\alpha_j\xi\hat{a})\exp(\mathrm{i}\beta_j\xi^*\hat{a}^\dagger)], \tag{4.8}$$

where α_j, β_j are complex numbers such that $\alpha_1 + \cdots + \alpha_N = \beta_1 + \cdots + \beta_N = 1$. This condition ensures that (4.8) reduces to an identity when $\hat{a} \to \alpha$, a c-number. By applying (2.88) repeatedly, we may combine the product of the exponentials in (4.8) in to a single exponential. We note that each such combination would contribute a c-number exponential whose exponent is proportional to $|\xi|^2$. As a result, we may rewrite (4.8) as

$$\Pi_{j=1}^{N}[\exp(\mathrm{i}\alpha_j\xi\hat{a})\exp(\mathrm{i}\beta_j\xi^*\hat{a}^\dagger)] = \exp\left[-\frac{s}{2}|\xi|^2\right]\exp\left\{\mathrm{i}(\xi\hat{a} + \xi^*\hat{a}^\dagger)\right\}. \tag{4.9}$$

Here s is a complex number related with products of the $\alpha's$ and the $\beta's$. Although the exact expression of s in terms of the $\alpha's$ and the $\beta's$ may be derived, it is inessential. The ordering for $s = 0$ is called the *Weyl ordering*. For the reason mentioned after (4.12), it is also called the *symmetric ordering*. In applications, it is often useful to know the form of operators in the normal or antinormal ordering introduced in Sect. 2.4. By applying (2.88), the exponential operator in (4.9) may be put in the antinormal or the normal ordering as

$$\begin{aligned}\exp\left\{\mathrm{i}(\xi\hat{a} + \xi^*\hat{a}^\dagger)\right\} &= \exp\left(\frac{1}{2}|\xi|^2\right)\exp(\mathrm{i}\xi\hat{a})\exp(\mathrm{i}\xi^*\hat{a}^\dagger) \\ &= \exp\left(-\frac{1}{2}|\xi|^2\right)\exp(\mathrm{i}\xi^*\hat{a}^\dagger)\exp(\mathrm{i}\xi\hat{a}).\end{aligned} \tag{4.10}$$

The ordering corresponding to different choices of the $\alpha's$ and the $\beta's$ in (4.8) thus reduces to the one defined in terms of just a complex number s. It is referred to as the *s-ordering* [56]. The operator ordering corresponding to a c-number form $\alpha^m\alpha^{*n}$ in the s-ordering may be derived by noting that

$$\begin{aligned}&\alpha^m\alpha^{*n} \to \Omega_s\left(\hat{a}^m\hat{a}^{\dagger n}\right) \\ &= \frac{\partial^m}{\partial(\mathrm{i}\xi)^m}\frac{\partial^m}{\partial(\mathrm{i}\xi^*)^n}\exp\left(-\frac{s}{2}|\xi|^2\right)\exp\left\{\mathrm{i}(\xi\hat{a} + \xi^*\hat{a}^\dagger)\right\}\Big|_{\xi=\xi^*=0}.\end{aligned} \tag{4.11}$$

As an example, the s-ordering of the powers of the operators in terms of the normal ordering is obtained by expressing the exponential operator in (4.11) in the normal-ordered form using (4.10) and then carrying the operation of differentiation. We leave it to the reader to show that

$$\alpha^m \alpha^{*n} \to \sum_{k=0} \left[\left(\frac{s+1}{2} \right)^k \frac{m!n!}{(m-k)!(n-k)!k!} \hat{a}^{\dagger n-k} \hat{a}^{m-k} \right]. \tag{4.12}$$

It may be shown that if $s = 0$ then the operator on the right hand side above may be expressed as a symmetric combination (see [57] for a proof). In the symmetric ordering, a product $x_1 x_2 \ldots x_m$ of variables is replaced by the sum of all possible permutations of the product divided by the total number of such permutations.

The QPD in the s-ordering, obtained by substituting (4.9), read with (4.8), in (4.7) reads

$$f^{(s)}(\alpha, \alpha^*) = \frac{1}{\pi^2} \int \mathrm{d}^2\xi \Big[\exp\{\mathrm{i}(\alpha\xi + \alpha^*\xi^*)\} \times \exp\left(-\frac{s}{2}|\xi|^2\right) \mathrm{Tr}\left[\exp\left\{-\mathrm{i}(\hat{a}\xi + \hat{a}^\dagger\xi^*)\right\}\hat{\rho}\right] \Big]. \tag{4.13}$$

This relation may be inverted by using (A.10) and (A.11) to express $\hat{\rho}$ in terms of the QPD as

$$\mathrm{Tr}\left[\exp\left\{-\mathrm{i}(\hat{a}\xi + \hat{a}^\dagger\xi^*)\right\}\hat{\rho}\right] = G(\xi, \xi^*) \exp\left(\frac{s}{2}|\xi|^2\right), \tag{4.14}$$

$$G(\xi, \xi^*) = \int \mathrm{d}^2\alpha \; f^{(s)}(\alpha, \alpha^*) \exp\{-\mathrm{i}(\alpha\xi + \alpha^*\xi^*)\}. \tag{4.15}$$

Now, disentangle the exponential under the trace in (4.14) into the antinormal form using (4.10). Use the relation

$$\mathrm{Tr}\left[\exp(-\mathrm{i}\xi\hat{a})\exp(-\mathrm{i}\xi^*\hat{a}^\dagger)\hat{\rho}\right] = \mathrm{Tr}\left[\exp(-\mathrm{i}\xi\hat{a}^\dagger)\hat{\rho}\exp(-\mathrm{i}\xi^*\hat{a})\right] \tag{4.16}$$

to carry the operation of trace in the resulting expression by applying (3.30). It will be found that

$$\frac{1}{\pi} \int \mathrm{d}^2\beta \exp\{-\mathrm{i}(\beta\xi + \beta^*\xi^*)\} \langle\beta|\hat{\rho}|\beta\rangle = G(\xi, \xi^*) \exp\left\{-\frac{1-s}{2}|\xi|^2\right\}. \tag{4.17}$$

This, on applying (A.10) and (A.11), leads to the expression

$$\langle\beta|\hat{\rho}|\beta\rangle = \frac{1}{\pi} \int \mathrm{d}^2\xi \; G(\xi, \xi^*) \exp\{\mathrm{i}(\beta\xi + \beta^*\xi^*)\} \times \exp\left\{-\frac{1-s}{2}|\xi|^2\right\} \tag{4.18}$$

for the density matrix in the coherent state representation. On invoking (3.32) and (3.33), the operator form of (4.18) is found to read

$$\hat{\rho} = \frac{1}{\pi}\int \mathrm{d}^2\xi\; G(\xi,\xi^*)\exp\left[-\frac{1-s}{2}|\xi|^2\right]\exp(\mathrm{i}\hat{a}^\dagger\xi^*)\exp(\mathrm{i}\hat{a}\xi)$$
$$= \frac{1}{\pi}\int \mathrm{d}^2\xi\; G(\xi,\xi^*)\exp\left(\frac{s}{2}|\xi|^2\right)\exp[\mathrm{i}(\hat{a}\xi+\hat{a}^\dagger\xi^*)]$$
$$= \frac{1}{\pi}\int \mathrm{d}^2\xi\; G(\xi,\xi^*)\exp\left[\frac{s+1}{2}|\xi|^2\right]\exp(\mathrm{i}\hat{a}\xi)\exp(\mathrm{i}\hat{a}^\dagger\xi^*). \tag{4.19}$$

This determines the relationship between the density operator and its various phase space representations through $G(\xi,\xi^*)$ defined in (4.15).

The relationship between different phase space representatives $f^{(s)}$ and $f^{(t)}$ may be derived by substituting (4.14) and (4.15), with s replaced by t, in (4.13) and on carrying the ξ-integration we find that, provided $\mathrm{Re}(s) > \mathrm{Re}(t)$,

$$f^{(s)}(\alpha,\alpha^*) = \frac{2}{\pi(s-t)}\int \mathrm{d}^2\beta\exp\left[-\frac{2|\alpha-\beta|^2}{s-t}\right] f^{(t)}(\beta,\beta^*). \tag{4.20}$$

This is the desired relation between two phase space distributions.

In practical applications, we need to convert equations involving products of $\hat{\rho}$ with $\hat{a}^m\hat{a}^{\dagger n}$ in to c-number equations (see Chaps.13,14). The c-number equivalent of $\hat{a}^m\hat{\rho}$ is obtained by replacing $\hat{\rho}$ in (4.13) by $\hat{a}^m\hat{\rho}$. On applying (4.10) and using the cyclic property of the trace, we note that

$$\mathrm{Tr}\left[\exp\left\{-\mathrm{i}(\hat{a}\xi+\hat{a}^\dagger\xi^*)\right\}\hat{a}^m\hat{\rho}\right]$$
$$= \exp(-|\xi|^2/2)\mathrm{Tr}\left[\exp(-\mathrm{i}\hat{a}\xi)\hat{a}^m\hat{\rho}\exp(-\mathrm{i}\hat{a}^\dagger\xi^*)\right]$$
$$= \exp(-|\xi|^2/2)\frac{\partial^m}{\partial(-\mathrm{i}\xi)^m}\mathrm{Tr}\left[\exp(-\mathrm{i}\hat{a}\xi)\hat{\rho}\exp(-\mathrm{i}\hat{a}^\dagger\xi^*)\right]. \tag{4.21}$$

Substitute this in (4.13) to obtain

$$\hat{a}^m\hat{\rho} \to \frac{1}{\pi^2}\int \mathrm{d}^2\xi\left[\alpha+\mathrm{i}\frac{s+1}{2}\xi^*\right]^m\left[\exp\{\mathrm{i}(\alpha\xi+\alpha^*\xi^*\}\right.$$
$$\left.\times\exp\left\{-\frac{s}{2}|\xi|^2\right\}\mathrm{Tr}\left[\exp\left\{-\mathrm{i}(\hat{a}\xi+\hat{a}^\dagger\xi^*)\right\}\hat{\rho}\right]\right]$$
$$= \left[\alpha+\frac{s+1}{2}\frac{\partial}{\partial\alpha^*}\right]^m f^{(s)}(\alpha,\alpha^*), \tag{4.22}$$

the $f^{(s)}(\alpha,\alpha^*)$ being the QPD of $\hat{\rho}$. The phase space equivalent of $\hat{a}^{\dagger m}\hat{\rho}$ may be derived in a similar manner but by starting with the anti-normal form of the exponential operator under the trace in (4.13). Verify that

$$\hat{a}^{\dagger m}\hat{\rho} \to \left[\alpha^*+\frac{s-1}{2}\frac{\partial}{\partial\alpha}\right]^m f^{(s)}(\alpha,\alpha^*). \tag{4.23}$$

The relations (4.22), (4.23) and their hermitian conjugates may be used to find the phase space equivalent of any combination $f_1(\hat{a},\hat{a}^\dagger)\hat{\rho}f_2(\hat{a},\hat{a}^\dagger)$ in terms of a differential operator on the phase space representative of $\hat{\rho}$.

Operator Averages. The foregoing considerations for a density operator $\hat{\rho}$ may be extended to construct the phase space representation $A^{(s)}(\alpha, \alpha^*)$ of any operator $\hat{A}$ by way of the correspondence $\hat{\rho} \to \hat{A}$ and $f^{(s)}(\alpha, \alpha^*) \to A^{(s)}(\alpha, \alpha^*)$ in (4.13) so that

$$A^{(s)}(\alpha, \alpha^*) = \frac{1}{\pi^2} \int \mathrm{d}^2\xi \Bigg[\exp\{\mathrm{i}(\alpha\xi + \alpha^*\xi^*)\} \times \exp\left\{-\frac{s}{2}|\xi|^2\right\} \mathrm{Tr}\left[\exp\{-\mathrm{i}(\hat{a}\xi + \hat{a}^\dagger\xi^*)\}\, \hat{A}\right]\Bigg]. \tag{4.24}$$

The expression for $\hat{A}$ in terms of its phase space representative, obtained similarly using (4.19), reads

$$\hat{A} = \frac{1}{\pi} \int \mathrm{d}^2\xi\; G_A(\xi, \xi^*) \exp\left(\frac{s}{2}|\xi|^2\right) \exp\left[\mathrm{i}(\hat{a}\xi + \hat{a}^\dagger\xi^*)\right] \tag{4.25}$$

with

$$G_A(\xi, \xi^*) = \int \mathrm{d}^2\alpha\; A^{(s)}(\alpha, \alpha^*) \exp\{-\mathrm{i}(\alpha\xi + \alpha^*\xi^*)\}. \tag{4.26}$$

The average of $\hat{A}$ may now be expressed in terms of a phase space integral similar in appearance to the classical expression (4.1). To that end, multiply (4.19) by $\hat{A}$, substitute for $G(\xi, \xi^*)$ from (4.15) and take its trace to obtain

$$\begin{aligned} \mathrm{Tr}\left[\hat{A}\hat{\rho}\right] &= \frac{1}{\pi} \int \mathrm{d}^2\xi \mathrm{d}^2\alpha\; f_s(\alpha, \alpha^*) \exp\left[-\mathrm{i}(\alpha\xi + \alpha^*\xi^*\right] \exp\left(\frac{s}{2}|\xi|^2\right) \\ &\quad \times \mathrm{Tr}\left[\hat{A} \exp\{\mathrm{i}(\hat{a}\xi + \hat{a}^\dagger\xi^*\}\right] \\ &= \pi \int \mathrm{d}^2\alpha\;\; f^{(s)}(\alpha, \alpha^*)\, A^{(-s)}(\alpha, \alpha^*). \end{aligned} \tag{4.27}$$

In writing the last line above, we have used the definition (4.24). The expression (4.27) is evidently of the form (4.1) of a classical average. The function $A^{(-s)}(\alpha, \alpha^*)$ is said to be *conjugate* to $A^{(s)}(\alpha, \alpha^*)$. We thus see that the expectation value of an operator is the phase space integral of the product of any of its phase space function with its conjugate representative of the density operator.

Next, we list some properties of the phase space representation for some particular values of s of special interest.

1. Let

$$\hat{\rho} = \int \mathrm{d}^2\beta\; P(\beta, \beta^*)\, |\beta\rangle\langle\beta|. \tag{4.28}$$

This is known as the P-function representation of a density operator. Substitute this in (4.13) and express the exponential operator under the trace there in the normal-ordering. Let $s = -1$. Use the cyclic property of the trace to obtain

$$
\begin{aligned}
& f^{(-1)}(\alpha, \alpha^*) \\
& = \frac{1}{\pi^2} \int \mathrm{d}^2\xi \int \mathrm{d}^2\beta \Big[P(\beta, \beta^*) \exp\{\mathrm{i}(\alpha\xi + \alpha^*\xi^*)\} \\
& \quad \times \mathrm{Tr}\Big[\exp(-\mathrm{i}\hat{a}\xi) |\beta\rangle\langle\beta| \exp(-\mathrm{i}\hat{a}^\dagger\xi^*) \Big] \Big] \\
& = \frac{1}{\pi^2} \int \mathrm{d}^2\xi \int \mathrm{d}^2\beta P(\beta, \beta^*) \exp[\mathrm{i}\{(\alpha - \beta)\xi + (\alpha^* - \beta^*)\xi^*\}] \\
& = P(\alpha, \alpha^*) . \qquad (4.29)
\end{aligned}
$$

The phase space representative for $s = -1$ is thus the P-function. The P-function representation for an antinormally ordered product can be found easily by using the completeness relation (3.29). For, on operating that relation with $\hat{a}^n$ on the left and $\hat{a}^{\dagger m}$ on the right we find that

$$
\hat{a}^n \hat{a}^{\dagger m} = \frac{1}{\pi} \int \mathrm{d}^2\alpha \; \alpha^n \alpha^{*m} |\alpha\rangle\langle\alpha| . \qquad (4.30)
$$

On comparing this with (4.28) we see that the P-function for $\hat{a}^n\hat{a}^{\dagger m}$ is $\alpha^n\alpha^{*m}/\pi$.

2. Next, set $s = 1$ in (4.17) and compare it with (4.15) to note that

$$
\frac{1}{\pi}\langle\beta|\hat{\rho}|\beta\rangle = f^{(1)}(\beta, \beta^*) \equiv Q(\beta, \beta^*) . \qquad (4.31)
$$

Thus $f^{(s)}(\beta, \beta^*)$ for $s = 1$ is simply the matrix element of the operator in the coherent states representation. It is also known as the Q-function or the *Husimi function*. The Q-function of $\hat{a}^{\dagger m}\hat{a}^n$ is clearly $\alpha^{*m}\alpha^n/\pi$. As a consequence of (4.29), (4.31) and (4.27), it follows that the trace of a product of two operators is the phase space integral of the P-function of one with the Q-function of the other. Thus, if the density operator is represented by its P-function then

$$
\int \mathrm{d}^2\beta P(\beta, \beta^*) \beta^{*m}\beta^n = \langle \hat{a}^{\dagger m}\hat{a}^n \rangle . \qquad (4.32)
$$

This shows that the phase space average of $\beta^{*m}\beta^n$ with the P-function of the density operator gives the average of normally ordered operators. On the other hand, if the density operator is represented by its Q-function then

$$
\int \mathrm{d}^2\beta Q(\beta, \beta^*) \beta^{*m}\beta^n = \langle \hat{a}^n\hat{a}^{\dagger m} \rangle . \qquad (4.33)
$$

This shows that the phase space average of $\beta^{*m}\beta^n$ with the Q-function of the density operator gives the average of antinormally ordered operators.

3. The phase space distribution function $f^{(s)}(\beta, \beta^*)$ corresponding to $s = 0$ is called the *Wigner function*. It is usually denoted by $W(\beta, \beta^*) \equiv f^{(0)}(\beta, \beta^*)$. We infer from (4.27) that the trace of a product of two operators is the phase space integral of the product of their Wigner functions.

Hence, if the density operator is represented in terms of its Wigner function then

$$\int \mathrm{d}^2\beta W(\beta,\beta^*)\,\beta^{*m}\beta^n = \frac{1}{N_s}\langle\left(\hat{a}^n\hat{a}^{\dagger m}\right)_s\rangle, \tag{4.34}$$

the suffix s denotes symmeterized operator product, i.e. the sum of all products formed by permutation of m $\hat{a}^\dagger$'s and n $\hat{a}$'s and N_s is the number of such permutations.

The relation (4.9) for an operator representation of a c-number function may be generalized by replacing $\hat{a}$ by a linear combination $\hat{b} = \eta_1\hat{a} + \eta_2\hat{a}^\dagger$:

$$\Omega\left\{\exp(\mathrm{i}\xi\hat{a})\exp(\mathrm{i}\xi^*\hat{a}^\dagger)\right\} = \Pi_{j=1}^N[\exp(\mathrm{i}\alpha_j\xi\hat{b})\exp(\mathrm{i}\beta_j\xi^*\hat{b}^\dagger)]. \tag{4.35}$$

The phase space distribution functions for such a generalized rule can be constructed by following the procedure outlined above for the special case $\eta_1 = 1, \eta_2 = 0$. The theory of quasi-distributions of the canonical operators is developed in its generality in [58].

4.2 Phase Space Representation of Spins

Consider a system of spins. The spin observables obey non canonical commutation relations. As shown in Chap. 2, we can represent spin operators as bilinear combinations of two canonical operators and thus extend the considerations of the Sect. 4.1 to construct the QPD for a spin [59]. However, a direct approach is to exploit the fact that a spin-S traverses the surface of a sphere. Hence, a spin may be described classically by a distribution function $f(\theta,\phi)$ of the polar and the azimuthal angles. In order to construct its quantum analog, $f(\theta,\phi)$ is expressed in terms of the averages of a complete set of functions. A convenient set in this case is the set of spherical harmonics $Y_{LM}(\theta,\phi)$, (L=0,1,...), (M=-L,-L+1,...,L) so that

$$\begin{aligned} f(\theta,\phi) &= \int \sin(\theta')\mathrm{d}\theta'\mathrm{d}\phi'\delta(\phi-\phi')\delta(\cos(\theta)-\cos(\theta'))f(\theta',\phi') \\ &= \int \sin(\theta')\mathrm{d}\theta'\mathrm{d}\phi' \sum_{L=0}^{\infty}\sum_{M=-L}^{L} Y_{LM}(\theta,\phi)Y_{LM}^*(\theta',\phi')f(\theta',\phi') \\ &= \sum_{L=0}^{\infty}\sum_{M=-L}^{L} Y_{LM}(\theta,\phi)\left\langle Y_{LM}^*(\theta,\phi)\right\rangle . \end{aligned} \tag{4.36}$$

In writing the second line above we have invoked the completeness relation (A.37) of the spherical harmonics. The QPD for a system of spins is obtained by replacing the classical average of the spherical harmonics in this equation by the quantum mechanical expectation value of appropriate operators. The operators appropriate for this purpose evidently are the ones which transform

under rotation in the same way as do the spherical harmonics. Now, recall that the operators corresponding to an integral spin may be represented by differential operators in θ, ϕ on the functions of θ, ϕ. We know the commutation relations between integral spin operators in the (θ, ϕ) representation and the $Y_{LM}(\theta, \phi)$ [60]. The defining property of the operators we are looking for corresponding to $Y_{LM}(\theta, \phi)$ in (4.36) is that their commutation relations with the spin operators be the same as the those between the spherical harmonics and the spin operators in the (θ, ϕ) representation. Hence, the desired operators should have the form $\hat{T}_{KQ}$ with $K = 0, 1, \ldots$ and $Q = -K, -K+1, \ldots, K$ and they should be such that

$$\begin{aligned} \left[\hat{S}_z,\ \hat{T}_{KQ}\right] &= Q\hat{T}_{KQ}, \\ \left[\hat{S}_\pm,\ \hat{T}_{KQ}\right] &= \sqrt{(K \mp Q)(K \pm Q + 1)}\hat{T}_{K\pm 1Q}. \end{aligned} \tag{4.37}$$

These are the commutation relations of $Y_{KQ}(\theta, \phi)$ with the spin operators in the (θ, ϕ) representation. We recall that, for a system of total spin quantum number S [60],

$$\hat{T}_{KQ} = \sum_{m,n=-S}^{S} (-)^{S-m}(-)^{j-n} C^{SSK}_{m-nQ} |m, S\rangle\langle n, S|, \tag{4.38}$$

$K = 0, 1, \ldots, 2S$; $Q = -K, -K+1, \ldots, K$. Substitution of (4.38) in (4.37) shows that C^{SSK}_{m-nQ} obey the recursion relation of *Wigner or Clebsch-Gordan coefficients*[60]. The operators $\hat{T}_{KQ}$, called *state multipole operators*, constitute a complete set. Their orthonormality relation is [60]

$$\mathrm{Tr}\left[\hat{T}^\dagger_{LM}\hat{T}_{KQ}\right] = \delta_{KL}\delta_{MQ}. \tag{4.39}$$

We note also the property

$$\hat{T}^\dagger_{KQ} = (-)^Q \hat{T}_{K-Q}. \tag{4.40}$$

Hence, any spin operator, for example, a density operator, may be expressed as

$$\hat{\rho} = \sum_{K=0}^{2S} \sum_{Q=-K}^{K} \left\langle \hat{T}^\dagger_{KQ} \right\rangle \hat{T}_{KQ}. \tag{4.41}$$

Now, the QPD of spins is obtained by identifying the classical average of $Y_{LM}(\theta, \phi)$ over (θ, ϕ) as an average of $\hat{T}_{LM}$ in the state described by $\hat{\rho}$ by means of the relation

$$\langle Y_{LM}(\theta, \phi)\rangle = \Omega_{LM}\mathrm{Tr}\left[\hat{T}_{LM}\hat{\rho}\right] \equiv \Omega_{LM}\left\langle \hat{T}_{LM} \right\rangle, \tag{4.42}$$

Ω_{LM} being a free constant. Substitution of this in (4.36) results in different quasiprobability distribution functions

$$f^{(\Omega)}(\theta,\phi)=\sum_{L=0}^{2S}\sum_{M=-L}^{L}Y_{LM}(\theta,\phi)\Omega_{LM}\mathrm{Tr}\left[T_{LM}^{\dagger}\hat{\rho}\right], \tag{4.43}$$

for different choices of the value of Ω_{LM}. By using the orthogonality relation (A.36) of the spherical harmonics, (4.43) may be inverted to obtain

$$\mathrm{Tr}\left[T_{LM}^{\dagger}\hat{\rho}\right]=\Omega_{LM}^{-1}\int\sin(\theta)\mathrm{d}\theta\mathrm{d}\phi f^{(\Omega)}(\theta,\phi)Y_{LM}^{*}(\theta,\phi). \tag{4.44}$$

Substitution of this in (4.41) expresses $\hat{\rho}$ in terms of the $f^{(\Omega)}(\theta,\phi)$.

The phase space distribution (4.43) is not normalized. We normalize it by noting that

$$\int\sin(\theta)\mathrm{d}\theta\mathrm{d}\phi f^{(\Omega)}(\theta,\phi)=\sqrt{\frac{4\pi}{2S+1}}\Omega_{00}. \tag{4.45}$$

In arriving at this result, we have inserted (4.43) for $f^{(\Omega)}(\theta,\phi)$ and carried the integration using (A.36) by setting $K=Q=0$ in it, and used (A.35) for $Y_{00}(\theta,\phi)$ along with the relation

$$\hat{T}_{00}=\frac{1}{\sqrt{2S+1}}. \tag{4.46}$$

Now, in analogy with (4.43), we may define the phase space representation $A^{(\Omega)}(\theta,\phi)$ of any spin operator $\hat{A}$ as

$$A^{(\Omega)}(\theta,\phi)=\sum_{L=0}^{\infty}\sum_{M=-L}^{L}Y_{LM}(\theta,\phi)\Omega_{LM}\mathrm{Tr}\left[T_{LM}^{\dagger}\hat{A}\right]. \tag{4.47}$$

The function $A^{(\Omega)}(\theta,\phi)$ determines $\hat{A}$ through the relations

$$\hat{A}=\sum_{L=0}^{2S}\sum_{M=-L}^{L}\mathrm{Tr}\left[\hat{T}_{LM}^{\dagger}\hat{A}\right]\hat{T}_{LM}, \tag{4.48a}$$

$$\mathrm{Tr}\left[\hat{T}_{LM}^{\dagger}\hat{A}\right]=\Omega_{LM}^{-1}\int\sin(\theta)\mathrm{d}\theta\mathrm{d}\phi\ A^{(\Omega)}(\theta,\phi)Y_{LM}^{*}(\theta,\phi). \tag{4.48b}$$

Next, we express the trace of a product of two operators in terms of their phase space representation. To that end, multiply (4.48a) by $\hat{B}$ and take the trace to get

$$\begin{aligned}\mathrm{Tr}\left[\hat{A}\hat{B}\right]&=\sum_{L,M}\mathrm{Tr}\left[T_{LM}^{\dagger}\hat{A}\right]\mathrm{Tr}\left[T_{LM}\hat{B}\right]\\&=\sum_{L,M}(-)^{M}\mathrm{Tr}\left[T_{LM}^{\dagger}\hat{A}\right]\mathrm{Tr}\left[T_{L-M}^{\dagger}\hat{B}\right].\end{aligned} \tag{4.49}$$

We have applied (4.40) in writing the second line above. Use (4.48b) and the property (A.34) to reduce (4.49) to the form

$$\mathrm{Tr}\left[\hat{A}\hat{B}\right] = \int \sin(\theta)\mathrm{d}\theta\mathrm{d}\phi \int \sin(\theta')\mathrm{d}\theta'\mathrm{d}\phi' \sum_{L,M} \frac{1}{\Omega_{LM}\Omega'_{L\ -M}} \times\left[Y_{LM}(\theta,\phi)Y^{*}_{LM}(\theta',\phi')A^{(\Omega)}(\theta',\phi')B^{(\Omega')}(\theta,\phi)\right]. \tag{4.50}$$

If $\Omega_{LM}\Omega'_{L\ -M} = 1$ then an application of the completeness relation (A.37) to (4.50) leads to

$$\mathrm{Tr}\left[\hat{A}\hat{B}\right] = \int \sin(\theta)\mathrm{d}\theta\mathrm{d}\phi A^{(\Omega)}(\theta,\phi)B^{(\tilde{\Omega})}(\theta,\phi), \tag{4.51}$$

where $B^{(\tilde{\Omega})}$ is the phase space functions corresponding to $\tilde{\Omega}_{LM} = 1/\Omega_{L\ -M}$. The functions $B^{(\Omega)}$ and $B^{(\tilde{\Omega})}$ are said to be *conjugate* to each other. The equation (4.51) shows that the trace of the product of two operators is the phase space integral of the phase space representation of one with the conjugate representation of the other. In particular, if one of the operators, say, $\hat{A}$ is the density operator then (4.51) determines the quantum expectation value of $\hat{B}$ in terms of the phase space integral. Now, in analogy with the considerations of the last section, let

$$\hat{\rho} = \int \sin(\theta)\mathrm{d}\theta d\phi P(\theta,\phi)|\theta,\phi\rangle\langle\theta,\phi|, \tag{4.52}$$

be the P-function representation. It implies that

$$\langle\hat{T}^{\dagger}_{LM}\rangle = \int \sin(\theta)\mathrm{d}\theta\mathrm{d}\phi P(\theta,\phi)\langle\theta,\phi|\hat{T}^{\dagger}_{LM}|\theta,\phi\rangle. \tag{4.53}$$

It is known that [61]

$$\langle\theta,\phi|\hat{T}^{\dagger}_{LM}|\theta,\phi\rangle = f_{LM}Y^{*}_{LM}(\theta,\phi) \tag{4.54}$$

where

$$f_{LM} = (-)^{L-M}\sqrt{4\pi}\frac{(2S)!}{\sqrt{(2S-L)!(2S+L+1)!}}. \tag{4.55}$$

Substitute (4.54) in (4.53) and insert the resulting expression in (4.43). Set $\Omega_{LM} = \Omega^{(P)}_{LM}$ with

$$\Omega^{(P)}_{LM} = f^{-1}_{LM}. \tag{4.56}$$

The summation over L, M then is simply the completeness relation (A.37). It then follows that

$$f^{(\Omega)} = P(\theta,\phi). \tag{4.57}$$

Thus, the QPD corresponding to the Ω_{LM} given by (4.56) is the P-function. Verify that the normalization factor (4.45) in this case is unity.

Next, take the matrix element of (4.41) in the spin coherent state to obtain

$$\begin{aligned}\langle\theta,\phi|\hat{\rho}|\theta,\phi\rangle &= \sum_{K=0}^{2S}\sum_{Q=-K}^{K}\left\langle\hat{T}_{KQ}^{\dagger}\right\rangle\left\langle\theta,\phi|\hat{T}_{KQ}|\theta,\phi\right\rangle \\ &= \sum_{K=0}^{2S}\sum_{Q=-K}^{K}\Big[\Omega_{KQ}^{-1}f_{KQ}\int\sin(\theta')\mathrm{d}\theta'\mathrm{d}\phi' \\ &\quad\times f^{(\Omega)}(\theta',\phi')Y_{KQ}^{*}(\theta',\phi')Y_{KQ}(\theta,\phi)\Big]. \end{aligned} \tag{4.58}$$

In writing the second line above we have invoked (4.44) and (4.54). Now, if we let $\Omega_{LM}=\Omega_{LM}^{(Q)}$ with

$$\Omega_{LM}^{(Q)}=f_{LM} \tag{4.59}$$

then the summation in (4.58) reduces to the completeness relation (A.37) leading to the identification

$$f^{(\Omega)}(\theta,\phi)=\langle\theta,\phi|\hat{\rho}|\theta,\phi\rangle. \tag{4.60}$$

Thus, the QPD for the Ω_{LM} corresponding to (4.59) is the diagonal matrix element of the density operator. The corresponding normalization factor may be evaluated by using (4.45). The normalized form of (4.60), called the Q-function is then defined by

$$Q(\theta,\phi)=\frac{2S+1}{4\pi}\langle\theta,\phi|\hat{\rho}|\theta,\phi\rangle. \tag{4.61}$$

Next, the Wigner function for spins is defined, following the deliberations of the last section, as the function which is its own conjugate i.e. the one corresponding to $\Omega_{LM}=1$:

$$\Omega_{LM}^{(W)}=1. \tag{4.62}$$

Let $f^{(1)}(\theta,\phi)$ be the corresponding QPD. Its normalization factor may be found using (4.45). The normalized form of the QPD corresponding to (4.62) reading

$$W(\theta,\phi)=\sqrt{\frac{2S+1}{4\pi}}f^{(1)}(\theta,\phi) \tag{4.63}$$

is called the Wigner function for spins.

As an example, we give explicit expressions for the Q, P and the Wigner function W for a spin-1/2. Using the expressions for the Clebsch-Gordan coefficients [60], (4.38) yields

$$\begin{aligned}\hat{T}_{00}&=\frac{1}{\sqrt{2}}, \qquad \hat{T}_{10}=\sqrt{2}\hat{S}_z, \\ \hat{T}_{11}&=-\hat{S}_+, \qquad \hat{T}_{1-1}=\hat{S}_-. \end{aligned} \tag{4.64}$$

Use of these and relevant expressions for $Y_{LM}(\theta,\phi)$ and Ω_{LM} give [61]

$$Q(\theta,\phi) = \frac{1}{4\pi}\left[1 - 2\langle \boldsymbol{S}\rangle.\boldsymbol{n}(-\theta,\phi)\right],$$
$$P(\theta,\phi) = \frac{1}{4\pi}\left[1 - 6\langle \boldsymbol{S}\rangle.\boldsymbol{n}(-\theta,\phi)\right],$$
$$W(\theta,\phi) = \frac{1}{4\pi}\left[1 + 2\sqrt{3}\langle \boldsymbol{S}\rangle.\boldsymbol{n}(\theta,\phi)\right], \tag{4.65}$$

the $\boldsymbol{n}(\theta,\phi) \equiv (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$ being the unit vector in the direction (θ,ϕ).

For a spin, it is possible to define, besides the phase space distributions introduced above, another class of quasiprobabilities. It is the distribution of the eigenvalues of its non-commuting components discussed next.

4.3 Quasiprobabilitiy Distributions for Eigenvalues of Spin Components

Any component of a spin-1/2 can assume two values, namely, $\pm 1/2$. Hence, we can construct a classical analog of a spin-1/2 by treating a spin component $S_a = \boldsymbol{S}.\boldsymbol{a}$ in any direction $\boldsymbol{a}$ as a two-state random variable capable of assuming the values $\pm 1/2$. The classical statistical description of such a system is provided by the probability $p_m(\epsilon_a, \epsilon_b, \ldots, \epsilon_m)$ that the spin components in the directions $\boldsymbol{a}, \boldsymbol{b}, \ldots, \boldsymbol{m}$ assume values $\epsilon_a/2, \epsilon_b/2, \ldots, \epsilon_m/2$ where $\epsilon_a, \epsilon_b, \ldots, \epsilon_m = \pm 1$. The probability distribution of the spin components $S_a, S_b, \ldots, S_m$ is then given by

$$f_m(S_a, \ldots, S_m) = \sum_{\{\epsilon\}} \delta\left(\frac{\epsilon_a}{2} - S_a\right) \cdots \delta\left(\frac{\epsilon_m}{2} - S_m\right) p_m(\{\epsilon\}), \tag{4.66}$$

where $\{\epsilon\} \equiv \epsilon_a, \ldots, \epsilon_m$. In the spirit of the approach developed in the last section, the quantum quasiprobabilities may be constructed from the classical distributions expressed in terms of the averages of the spin variables. To derive the expression for the classical distributions in terms of the averages of the spin variables, multiply (4.66) by $(S_a\alpha_a + 1/2)\cdots(S_m\alpha_m + 1/2)$, where $\alpha_a, \alpha_b, \ldots, \alpha_m = \pm 1$, and integrate over $S_a, S_b, \ldots, S_m$ to obtain

$$\left\langle \left(\frac{1}{2} + S_a\alpha_a\right)\left(\frac{1}{2} + S_b\alpha_b\right)\cdots\left(\frac{1}{2} + S_m\alpha_m\right)\right\rangle$$
$$= \frac{1}{2^m}\sum_{\{\epsilon\}} p_m(\{\epsilon\})(1 + \epsilon_a\alpha_a)(1 + \epsilon_b\alpha_b)....(1 + \epsilon_m\alpha_m). \tag{4.67}$$

The angular brackets denote the average with respect to $f_m(S_a, S_b, \cdots, S_m)$. Now, on noting that the possible values of $\epsilon_i\alpha_i$ are ± 1, it follows that the right hand side of (4.67) is non-zero only when $\epsilon_i\alpha_i = 1$ for all i so that

$$p_m(\{\epsilon\}) = \left\langle \left(\frac{1}{2} + S_a\epsilon_a\right)\left(\frac{1}{2} + S_b\epsilon_b\right)\cdots\left(\frac{1}{2} + S_m\epsilon_m\right)\right\rangle. \tag{4.68}$$

This is the desired expression for the probability distribution in terms of the averages.

The quantum analog of (4.68), constructed by replacing the c-number variables $S_a, S_b, \ldots, S_m$ by the operators $\hat{S}_a, \hat{S}_b, \ldots, \hat{S}_m$, may be written formally as

$$p_m^q(\{\epsilon\}) = \mathrm{Tr}\Big[\Big\{\Big(\frac{1}{2}+\hat{S}_a\epsilon_a\Big)\Big(\frac{1}{2}+\hat{S}_b\epsilon_b\Big)\cdots\Big(\frac{1}{2}+\hat{S}_m\epsilon_m\Big)\Big\}_{\mathrm{CO}}\hat{\rho}\Big]$$
$$\equiv \Big\langle\Big\{\Big(\frac{1}{2}+\hat{S}_a\epsilon_a\Big)\Big(\frac{1}{2}+\hat{S}_b\epsilon_b\Big)\cdots\Big(\frac{1}{2}+\hat{S}_m\epsilon_m\Big)\Big\}_{\mathrm{CO}}\Big\rangle, \quad (4.69)$$

where the suffix CO stands for 'chosen ordering' of the product of the operators and $\hat{\rho}$ is the quantum density operator of the spin. The $p_m^q(\{\epsilon\})$ is the joint quasiprobability distribution for the eigenvalues of the components of spin-1/2 in the directions $\boldsymbol{a}, \boldsymbol{b} \ldots, \boldsymbol{m}$.

The operator ordering problem, of course, does not arise if the distribution is sought only for one component, say, the component along a direction $\boldsymbol{a}$. The equation (4.69) then reads

$$p_1^q(\epsilon_a) = \mathrm{Tr}\Big[\Big(\frac{1}{2}+\hat{S}_a\epsilon_a\Big)\hat{\rho}\Big]. \quad (4.70)$$

Now, take the trace in this equation in the eigenstates $|\boldsymbol{a}, \pm 1/2\rangle$ of S_a to show that

$$p_1^q(+) = \Big\langle \boldsymbol{a}, \frac{1}{2}\Big|\hat{\rho}\Big|\boldsymbol{a}, \frac{1}{2}\Big\rangle. \quad (4.71)$$

Since $|\boldsymbol{a}, \frac{1}{2}\rangle$ is a spin coherent state (see Sect. 3.3), (4.71) is proportional to the Q function.

The quasiprobability for more than one component would depend upon the chosen ordering. Here we discuss only the completely symmetric ordering defined after (4.12). It turns out to be useful in formulating a criterion for a spin system to be labeled as classical or non classical. On using the anti commutation relation (1.130), the correspondence between some c-number and the operator products in symmetric ordering reads

$$S_aS_b \to \frac{1}{2}(\hat{S}_a\hat{S}_b + \hat{S}_a\hat{S}_b) = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{4}$$
$$S_aS_bS_c \to \frac{1}{12}\Big[(\hat{S}_a(\hat{S}_b\hat{S}_c + \hat{S}_c\hat{S}_b) + (\hat{S}_b\hat{S}_c + \hat{S}_c\hat{S}_b)\hat{S}_a$$
$$+(a\to b, b\to c, c\to a) + (a\to c, c\to b, b\to a)\Big]$$
$$= \frac{1}{6}\Big[\boldsymbol{a}\cdot\boldsymbol{b}\hat{S}_c + \boldsymbol{c}\cdot\boldsymbol{a}\hat{S}_b + \boldsymbol{b}\cdot\boldsymbol{c}\hat{S}_a\Big]. \quad (4.72)$$

Of particular interest is the quasiprobability for three orthogonal components. On using (4.72), the QPD (4.69) for three orthogonal components in the symmetric ordering reads

$$p_3^q(\epsilon_a, \epsilon_b, \epsilon_c) = \frac{1}{2^2}\ \mathrm{Tr}\Big[\Big\{\frac{1}{2} + \hat{S}_a\epsilon_a + \hat{S}_b\epsilon_b + \hat{S}_c\epsilon_c)\Big\}\hat{\rho}\Big]. \tag{4.73}$$

The preceding considerations can be generalized to a system of N spin-1/2s. The quasiprobability $p_m^q(\{\epsilon_{a^{(j)}}^{(j)}, \epsilon_{b^{(j)}}^{(j)}), \ldots, \epsilon_{m^{(j)}}^{(j)}\})$ that the components of the spins, labeled $1, 2 \ldots, N$, assume values $\{\epsilon_{a^{(j)}}^{(j)}, \epsilon_{b^{(j)}}^{(j)}, \ldots, \epsilon_{m^{(j)}}^{(j)}\}$ along the directions $\{\boldsymbol{a}^j, \boldsymbol{b}^j, \ldots, \boldsymbol{m}^j\}$ is given, on a straightforward generalization of (4.69), by

$$\begin{aligned} & p_m^q\left(\{\epsilon_{a^{(j)}}^{(j)}, \epsilon_{b^{(j)}}^{(j)}, \ldots, \epsilon_{m^{(j)}}^{(j)}\}\right) \\ & = \mathrm{Tr}\prod_{j=1}^{N}\left[\left\{\left(\frac{1}{2} + \hat{S}_{a^{(j)}}^{(j)}\epsilon_{a^{(j)}}^{(j)}\right)\cdots\left(\frac{1}{2} + \hat{S}_{m^{(j)}}^{(j)}\epsilon_{m^{(j)}}^{(j)}\right)\right\}_{\mathrm{CO}}\right]\hat{\rho} \\ & \equiv \left\langle \prod_{j=1}^{N}\left\{\left(\frac{1}{2} + \hat{S}_{a^{(j)}}^{(j)}\epsilon_{a^{(j)}}^{(j)}\right)\cdots\left(\frac{1}{2} + \hat{S}_{m^{(j)}}^{(j)}\epsilon_{m^{(j)}}^{(j)}\right)\right\}_{\mathrm{CO}}\right\rangle. \end{aligned} \tag{4.74}$$

The QPD for three orthogonal components in the symmetric ordering, obtained by generalizing (4.73), reads

$$\begin{aligned} & p_3^q(\{\epsilon_{a^{(j)}}^{(j)}, \epsilon_{b^{(j)}}^{(j)}, \epsilon_{c^{(j)}}^{(j)}\}) \\ & = \frac{1}{2^{2N}}\mathrm{Tr}\prod_{j=1}^{N}\left[\left(\frac{1}{2} + \hat{S}_{a^{(j)}}\epsilon_{a^{(j)}}^{(j)} + \hat{S}_{b^{(j)}}\epsilon_{b^{(j)}}^{(j)} + \hat{S}_{c^{(j)}}\epsilon_{c^{(j)}}^{(j)}\right)\right]\hat{\rho}. \end{aligned} \tag{4.75}$$

By following the steps leading from (4.70) to (4.71) and by invoking the fact that a spin coherent state for a system of N spin-1/2s is the product of the spin coherent state for each of the spins, it can be established that the Q-function for a system of N spin-1/2s is the same as $p_N^q(\{+\}_N)$.

We will see below that the QPDs introduced in this section play an important role in the scheme for identifying non classical states.

4.4 Classical and Non-classical States

In Chap. 1 we discussed the merits of the suggestion that the quantum indeterminism may be regarded as a classical statistical one arising from a lack of any knowledge about the dynamics of some hidden variables. States exhibiting properties not attributable to any classical statistical description are termed non-classical. Here we outline an approach for classifying the states as classical or non-classical based on the concept of quasiprobability distribution functions introduced earlier in this chapter.

4.4.1 Non-classical States of Electromagnetic Field

We showed in Sect. 4.1 that each QPD in phase space determines operator averages in certain order unique to it. We found that the QPD $f^{(s)}(\{q, p\})$

in s-ordering acts as a phase space distribution function for determining the averages of operators in the ordering $-s$ conjugate to s. If $f^{(s)}(\{q,p\})$ for a state is a classical distribution function, i.e. if it is normalizable and non-negative everywhere then the corresponding state is classical with respect to the measurement of operator averages in $-s$ ordering. This means that the moments of operators in $-s$ ordering in the said state will exhibit all the properties of the moments of a classical distribution. If $f^{(s)}(\{q,p\})$ assumes negative values or is non-normalizable then it is *non-classical*. The moments of operators in $-s$ ordering in that case may exhibit purely quantum characteristics, i.e. the properties not expected from the moments of a classical distribution. Such non classical characteristic are exhibited in the form of violation of some inequalities between the moments expected of a classical distribution.

Note that the Q-function, being the diagonal element of the density matrix, is always positive and normalizable. That function corresponds to $s = 1$ which characterizes the normal ordering. Hence every state is classical with respect to measurements of anti-normally ordered products corresponding to $s = -1$. The functions $f^{(s)}(\{q,p\})$ for other orderings may become non-classical. What value of s should one choose to classify the states? The answer to this question is provided by the mechanism by which experimental measurements are made. Here we confine our attention to the issue of characterization of the states of the electromagnetic (e.m.) field. We will see in Chap. 6 that the e.m. field is described by the harmonic oscillator operators and that the experimental measurements on it consist in measuring the expectation values of those operators in normal ordering. Since the normal order corresponds to $s = 1$, the QPD appropriate for classifying the states the e.m. field as classical or non-classical is the one corresponding to $s = -1$. As shown in (4.29), the QPD corresponding to $s = -1$ is the P-function. A state of the e.m. field is accordingly called classical if its P-function is classical. Else it is labeled non-classical.

In terms of this criterion, it can be proved that the coherent state is the only pure state of the e.m. field that is classical [62]. To that end, let $\hat{\rho}$ be the density operator of a pure state of the field and let $P(\alpha,\alpha^*)$ be the corresponding P-function. Since $\hat{\rho}$ describes a pure state, $\hat{\rho} = \hat{\rho}^2$. The normalizability demands that

$$\mathrm{Tr}\left[\hat{\rho}^2\right] = \mathrm{Tr}\left[\hat{\rho}\right] = \int \mathrm{d}^2\alpha\, P(\alpha,\alpha^*) = 1. \tag{4.76}$$

Now, the expression (4.28) of $\hat{\rho}$ in terms of $P(\alpha,\alpha^*)$ yields

$$\mathrm{Tr}\left[\hat{\rho}^2\right] = \int \mathrm{d}^2\alpha\, \mathrm{d}^2\beta\, \exp\left[-|\alpha-\beta|^2\right] P(\alpha,\alpha^*)P(\beta,\beta^*). \tag{4.77}$$

If $P(z,z^*)$ is a classical distribution function then it must be non-negative and normalizable as in (4.76). However, if that be so then the right hand side in (4.77) is always less than one unless

$$P(\alpha, \alpha^*) = \delta(\alpha - \alpha')\delta(\alpha^* - \alpha^{*\prime}). \tag{4.78}$$

Since, by virtue of (4.76), the left hand side of (4.77) is required to be always unity, it follows that (4.76) and (4.77) are consistent only if (4.78) holds which is the expression for the P-function of the coherent state $|\alpha'\rangle$.

The MUS may be shown to be the only pure states of the e.m. field whose Wigner function is non-negative [63].

The experimentally measurable quantities are, of course, the moments. Experimental observation of non-classicality of the P-function should, therefore, be translated in to the properties of the moments. Some such properties are discussed in Chap. 6.

4.4.2 Non-classical States of Spin-1/2s

In the absence of any guideline based on the method of measurement, other considerations come in to play for classifying the states of a system of spins as classical or non-classical. A consideration is suggested by the deliberations in Chap. 1 regarding non-classical characteristics of a pair of spin-1/2s. We found there that the non-classicality of a pair of spin-1/2s is linked with correlations between them. An uncorrelated state of spin-1/2s, and hence any state of a single spin-1/2, is classical. Recall from Chap. 3 that an uncorrelated pure state of a system of spins is its coherent state. We may, therefore, base the classification scheme on the premise that it should classify any uncorrelated state as classical. However, this scheme is not helped by the phase space distribution of a spin, introduced in the Sect. 4.2. For, (4.65) shows that the P and W functions are negative even for a spin in the eigenstate $|\pm 1/2\rangle$ which is a spin coherent state.

However, the QPD of the eigenvalues of spin components, introduced in Sect. 4.3, do lead to a scheme of classification of states according to the following criterion [25]:

A quantum state of a system of N spin-1/2s is classical if the joint quasiprobability for the eigenvalues of the components of each spin in three mutually orthogonal directions, one of which is the average direction of that spin, is classical in the symmetric ordering of the operators. It is non-classical if any of those m-spin ($m \leq N$) joint quasiprobabilities is negative in the said ordering.

This criterion identifies (i) the coherent states of N spin-1/2s as classical, (ii) any state of single spin-1/2 as classical; (iii) any pure entangled state of two spin-1/2s as non-classical: this is in conformity with the finding in Chap. 1 that any such state violates Bell's inequality which is a signature of non-classicality; (iv) squeezed spin states as non-classical. For details, see [25].

5. Theory of Stochastic Processes

In a number of situations, the forces acting on a system are non-deterministic. The dynamical variables of the system then are random functions of time. The behaviour of such variables can be described only statistically. The problem of studying the statistical behaviour of random functions of time is the subject of the theory of stochastic processes. In this chapter we summarize some concepts and the operational techniques of the theory of stochastic processes. For details, we refer to [64]- [67].

5.1 Probability Distributions

Consider a process described by a real valued function $\xi(t)$ of time t. If $\xi(t)$ traces unpredictable paths as a function of t in different realizations of an experiment then it is called a *random function* and the process described by it as a *stochastic process*. Let $\xi_i(t)$ be the functional form of the path traced by $\xi(t)$ in the i^{th} realization of an experiment. The set of all possible realizations $\xi_1(t), \xi_2(t), \ldots$ of $\xi(t)$ constitutes an ensemble of $\xi(t)$. The theory of stochastic processes deals with the problem of determining the ensemble–averaged values $\langle f(\xi(t_1), \cdots, \xi(t_n))\rangle$.

Consider first the problem of determining $\langle f(\xi(t))\rangle$. Let $p_1(x,t)\mathrm{d}x$ be the probability that the value of $\xi(t)$ at time t lies in the interval [x,x+dx]. The single time *probability density* (henceforth also called single time probability) $p_1(x,t)$ is defined by

$$p_1(x,t) = \langle \delta(x - \xi(t))\rangle \tag{5.1}$$

where the average is over all realizations of $\xi(t)$. The average of a function of $\xi(t)$ may be determined in terms of $p_1(x,t)$ by noting that

$$\langle f(\xi(t))\rangle = \langle \int f(x)\delta(x - \xi(t))\rangle \mathrm{d}x = \int p_1(x,t) f(x)\, \mathrm{d}x. \tag{5.2}$$

The integration is over the range of realizable values of $\xi(t)$. In a similar way, the average of a function of $\xi(t_1), \cdots, \xi(t_n)$ is determined by the n-time *joint probability density* (henceforth also called n-time probability) $p_n(x_n, t_n; \ldots; x_1, t_1) \equiv p_n(\{x_i, t_i\})$ defined by

$$p_n(\{x_i,t_i\}) = \langle \delta(x_1 - \xi(t_1)) \cdots \delta(x_n - \xi(t_n)) \rangle . \tag{5.3}$$

The $p_n(\{x_i,t_i\})\,\mathrm{d}x_1 \cdots \mathrm{d}x_n$ is the probability that $\xi(t)$ assumes values $\xi(t_1)$, $\ldots, \xi(t_n)$ at times $t_1, \cdots, t_n$ lying in the intervals $[x_1, x_1 + \mathrm{d}x_1], \ldots,$ $[x_n, x_n + \mathrm{d}x_n]$ respectively. The n-time averages are determined by the relation

$$\langle f(\xi(t_1), \ldots, \xi(t_n)) \rangle = \int \mathrm{d}x_1 \cdots \int \mathrm{d}x_n p_n(\{x_j,t_j\})\, f(x_1, \ldots, x_n) . \tag{5.4}$$

A stochastic process is completely described by the infinite hierarchy of the probability densities $p_n(\{x_i,t_i\})$, $(n = 1, 2, \ldots)$. However, note that

$$p_k(x_{n+k}, t_{n+k}; \cdots; x_n, t_n) = \int \mathrm{d}x_{n-1} \cdots \mathrm{d}x_1 p_{n+k}(\{x_j,t_j\}) . \tag{5.5}$$

This is the compatibility condition relating k-time probability with $n+k$-time probabilities.

A stochastic process is called *stationary* if its probability densities are invariant under time translation, i.e. if

$$p_n(\{x_i,t_i\}) = p_n(\{x_i,t_i+T\}), \tag{5.6}$$

where T is arbitrary. On setting $T = -t_1$ we see that

$$p_1(x,t) = p_1(x,0), \qquad p_2(x_1,t_1;x_2,t_2) = p_2(x_1,0;x_2,t_2-t_1). \tag{5.7}$$

This shows that single time probability of a stationary process is time-independent and that its two-time probability depends only on the time difference.

Now, by invoking the representation (A.1) of the delta function, (5.3) may be rewritten as

$$\begin{aligned} & p_n(\{x_i,t_i\}) \\ &= \left(\frac{1}{2\pi}\right)^n \int_{-\infty}^{\infty} \mathrm{d}u_1 \cdots \int_{-\infty}^{\infty} \mathrm{d}u_n C_n(\{u_i,t_i\}) \exp\left(-\mathrm{i}\sum_{i=1}^{n} x_i u_i\right), \end{aligned} \tag{5.8}$$

where the Fourier coefficients $C_n(\{u_i,t_i\})$ of $p_n(\{x_i,t_i\})$,

$$\begin{aligned} C_n(\{u_i,t_i\} &= \left\langle \exp\left(\mathrm{i}\sum_{k=1}^{n} u_k \xi(t_k)\right)\right\rangle \\ &= \sum_{m_1,\cdots,m_n=0}^{\infty} \frac{1}{m_1! \cdots m_n!} (\mathrm{i}u_1)^{m_1} \cdots (\mathrm{i}u_n)^{m_n} \langle \xi^{m_1}(t_1) \cdots \xi^{m_n}(t_n) \rangle \end{aligned} \tag{5.9}$$

is called the *characteristic function*. An average $\langle \xi^{m_1}(t_1) \cdots \xi^{m_n}(t_n) \rangle$ is called a *moment* of $\xi(t)$ and the number $m_1 + \cdots + m_n$ its order. The concept of characteristic function enables us to determine the probability densities in the problems specified in terms of the moments of a random function. Note from (5.9) that the characteristic function generates the moments by means of the relation

$$\left\langle \prod_{j=1}^{n} \xi^{m_j}(t_j) \right\rangle = \prod_{j=1}^{n} \left(\frac{\partial}{\partial \mathrm{i} u_j} \right)^{m_j} C_n\left(\{u_i, t_i\}\right) \Big|_{u_j=0} . \tag{5.10}$$

Now, if $\xi(t)$ assumes the same value at any time t in all its realizations without any correlation between its values at different times then any moment $\langle \xi^{m_1}(t_1) \cdots \xi^{m_n}(t_n) \rangle$ factorizes in to the product $\langle \xi(t_1) \rangle^{m_1} \cdots \langle \xi(t_n) \rangle^{m_n}$. We may classify the stochastic processes according to the deviation of its moments from their factorized value in terms of lower order moments. A measure of this deviation is provided by the *cumulants* $\langle\langle \xi^{m_1}(t_1) \cdots \xi^{m_n}(t_n) \rangle\rangle$ defined by

$$\langle\langle \xi^{m_1}(t_1) \cdots \xi^{m_n}(t_n) \rangle\rangle = \left[\prod_{j=1}^{n} \left(\frac{\partial}{\partial \mathrm{i} u_j} \right)^{m_j} \ln \left\{ C_n(\{u_i, t_i\}) \right\} \right]_{u_j=0} . \tag{5.11}$$

The number $m_1 + \cdots + m_n$ is the order of the cumulant in (5.11). This definition of cumulants is equivalent with the following expansion of the characteristic function in terms of the cumulants

$$\begin{aligned} & C_n\left(\{u_i, t_i\}\right) \\ &= \exp\left[\sum_{\{m_i=0\}}^{\infty} \frac{1}{m_1! \cdots m_n!} (\mathrm{i} u_1)^{m_1} \cdots (\mathrm{i} u_n)^{m_n} \langle\langle \xi^{m_1}(t_1) \cdots \xi^{m_n}(t_n) \rangle\rangle \right], \\ & m_1 + \cdots + m_n \neq 0. \end{aligned} \tag{5.12}$$

By expressing an integral as a sum followed by the use of (5.12), it may be shown that [64]

$$\begin{aligned} & \left\langle \exp\left[\mathrm{i} \int_0^t u(\tau) \xi(\tau) \mathrm{d}\tau \right] \right\rangle \\ &= \exp\left[\sum_{n=1}^{\infty} \frac{\mathrm{i}^n}{n!} \int_0^t \mathrm{d}\tau_1 \cdots \int_0^t \mathrm{d}\tau_n u(\tau_1) \cdots u(\tau_n) \right. \\ & \quad \left. \times \langle\langle \xi(\tau_1) \cdots \xi(\tau_n) \rangle\rangle \right]. \end{aligned} \tag{5.13}$$

It should be emphasized that the quantity inside a double angular bracket denoting a cumulant is not an algebraic operand. It is only a symbol indicating the highest moment in the expression of a cumulant in terms of the moments. Two lowest-order cumulants in terms of the moments are

$$\begin{aligned} \langle\langle \xi(t_i) \rangle\rangle &= \langle \xi(t_i) \rangle , \\ \langle\langle \xi(t_i) \xi(t_j) \rangle\rangle &= \langle \xi(t_i) \xi(t_j) \rangle - \langle \xi(t_i) \rangle \langle \xi(t_j) \rangle \equiv \sigma\left(t_i, t_j\right) . \end{aligned} \tag{5.14}$$

The first equation above shows that the first order cumulant is the mean whereas the second order cumulant is the variance if $t_j = t_i$ and a covariance if $t_j \neq t_i$. The covariance is a measure of correlation between the values of the random function at two different times. The elements $\sigma(t_i, t_j)$

$(i,j = 1,2,\cdots)$ constitute the *correlation matrix* $\hat{\sigma}$. It is a symmetric matrix. By virtue of the positivity of the probability densities, we can identify the average $\langle \xi(t_i)\xi(t_j)\rangle$ as a scalar product which fulfills the axioms of a scalar product listed in Sect. 1.1. Consequently, the generalized Schwarz's inequality (1.7) in this case reads $\det(\hat{\sigma}) \geq 0$ where the matrix $\hat{\sigma}$ has $\sigma(t_i,t_j)$ $(i,j = 1,2,\ldots,N)$ (N=1,2,...) as its elements. As an example, for $N = 2$ it states that $\sigma(t_i,t_i)\sigma(t_j,t_j) \geq \sigma^2(t_i,t_j)$.

Now, if all the moments of a process factorize in to the products of $\langle \xi(t_i)\rangle$ then $C_n(\{u\}) = \exp(\mathrm{i}u_1\langle\xi(t_1)\rangle)\cdots\exp(\mathrm{i}u_n\langle\xi(t_n)\rangle)$. Insert this in (5.11) and verify that $\langle\langle \xi^{m_1}(t_1)\cdots\xi^{m_n}(t_n)\rangle\rangle = 0$ if $m_1 + \cdots + m_n \geq 2$. This property characterizes a deterministic process. The cumulant expansion enables us to define the process next to a completely factorized one as the one for which $\langle\langle \xi^{m_1}(t_1)\cdots\xi^{m_n}(t_n)\rangle\rangle = 0$ if $m_1 + \cdots + m_n \geq 3$. It is called a *Gaussian process*. On expressing $\langle\langle \xi(t_i)\xi(t_j)\rangle\rangle$ in terms of $\sigma(t_i,t_j)$ as in (5.14), the characteristic function (5.12) for a Gaussian process assumes the form

$$C_n(\{u_i,t_i\}) = \exp\left(\mathrm{i}\sum_i u_i\langle\xi(t_i)\rangle - \frac{1}{2}\sum_{ij}\sigma(t_i,t_j)u_iu_j\right). \tag{5.15}$$

We can find the moments by substituting this in (5.10). As an example, let us determine single time moments assuming $\langle\xi(t_i)\rangle = 0$. To that end, expand $C_1(u_1)$ in (5.15) in powers of u and insert it in (5.10). Carry the required differentiation to show that single time moments of a Gaussian process are given by

$$\langle\xi^{2m}\rangle = \left(\frac{\sigma}{2}\right)^m\frac{(2m)!}{m!}, \quad \langle\xi^{2m+1}\rangle = 0. \tag{5.16}$$

Next, we derive the n-time probability distribution by inserting (5.15) in (5.8). We assume $\hat{\sigma}$ to be positive and carry the integration over $\{u_i\}$ by applying (A.22) to obtain

$$p_n(\{x_i,t_i\}) = \frac{1}{\sqrt{(2\pi)^n\det(\hat{\sigma})}}\exp\left[-\frac{1}{2}\left(\boldsymbol{X}^T - \langle\hat{\boldsymbol{\xi}}^T\rangle\right)\hat{\sigma}^{-1}\left(\boldsymbol{X} - \langle\hat{\boldsymbol{\xi}}\rangle\right)\right]. \tag{5.17}$$

Here $\boldsymbol{X}$ and $\boldsymbol{\xi}$ are columns formed by n elements $x_1,\ldots,x_n$ and $\xi(t_1),\ldots,\xi(t_n)$ respectively, and $\boldsymbol{A}^T$ denotes transpose of $\boldsymbol{A}$.

Next we outline the scheme for classifying stochastic processes according to their correlations in time. It leads to the concept of Markov processes which encompass a large variety of processes of practical interest.

5.2 Markov Processes

We noted in Sect. 5.1 that a stochastic process is characterized by the infinite hierarchy of the probability densities $p_n(\{x_i,t_i\})$. Of considerable importance

is also the concept of conditional probabilities. It enables us to classify the processes according to their correlations at different times

The probability density of finding $\xi(t)$ in the intervals $[x_r, x_r + \mathrm{d}x_r], \cdots [x_n, x_n + \mathrm{d}x_n]$ respectively at times $t_r, \ldots, t_n$ under the condition that $\xi(t)$ has known values $x_1, \ldots, x_{r-1}$ at times $t_1, \ldots, t_{r-1}$ $(t_1 < t_2 < \cdots < t_n)$ is called the *conditional probability density* $p(x_n, t_n; \ldots; x_r, t_r | x_{r-1}, t_{r-1}; \ldots; x_1, t_1)$. We note that, for $t_n > \cdots > t_1$,

$$p_n(\{x_i, t_i\}) = p(x_n, t_n | x_{n-1}, t_{n-1}; \ldots; x_1, t_1)\, p_{n-1}(\{x_i, t_i\}). \tag{5.18}$$

Consider the conditional probability $p(x_n, t_n | x_{n-1}, t_{n-1}; \ldots; x_1, t_1)$ for finding $\xi(t)$ in the interval $[x_n, x_n + \mathrm{d}x_n]$ at time t_n when its value at the earlier times $t_{n-1}, \ldots, t_1$ is known respectively to be $x_{n-1}, \ldots, x_1$. If this probability at any time t_n is independent of its values at the earlier times then

$$p(x_n, t_n | x_{n-1}, t_{n-1}; \ldots; x_1, t_1) = p_1(x_n, t_n). \tag{5.19}$$

On inserting this in (5.18) and on repeating the argument, it follows that

$$p_n(x_n, t_n; \ldots; x_1, t_1) = p_1(x_n, t_n) \cdots p_1(x_1, t_1). \tag{5.20}$$

This shows that there is no correlation between the values of the random function at different times. However, if $\xi(t)$ is a continuous function of time, we expect its value at a time to be correlated with its value at a time at least infinitesimally close to it in the past.

Bearing aforementioned arguments in mind, next in the scheme of classification based on the conditional probabilities would be the processes in which the value of the random function at a time t is correlated only with its value at the time immediately preceding t. Such a process, called a *Markov process*, is characterized by the relation

$$p(x_n, t_n | x_{n-1}, t_{n-1}; \ldots; x_1, t_1) = p(x_n, t_n | x_{n-1}, t_{n-1}), \tag{5.21}$$

$t_n > \cdots > t_1$. On substituting this in (5.18) we find that

$$p_n(\{x_i, t\}) = p(x_n, t_n | x_{n-1}, t_{n-1}) p_{n-1}(\{x_i, t\}) \tag{5.22}$$

On using (5.21) again and on repeating the argument, we obtain

$$\begin{aligned} &p_n(\{x_i, t_i\}) \\ &= p(x_n, t_n | x_{n-1}, t_{n-1}) \cdots p(x_2, t_2 | x_1, t_1) p_1(x_1, t_1). \end{aligned} \tag{5.23}$$

This implies that a Markov process is determined completely by its single time probability distribution $p_1(x, t)$ and the conditional probability $p(x_i, t_i | x_j, t_j)$ called the *transition probability*. The transition probability of a Markov process obeys the condition $(t_3 > t_2 > t_1)$

$$p(x_3, t_3 | x_1, t_1) = \int p(x_3, t_3 | x_2, t_2) p(x_2, t_2 | x_1, t_1) \mathrm{d}x_2 \tag{5.24}$$

called the *Chapman-Kolmogorov equation* [64]-[67]. The *rate of transition* from x_j to x_i for $x_i \neq x_j$ is defined by

$$w(x_i|x_j) = Lt_{\Delta t\to 0}\frac{p(x_i, t+\Delta t|x_j, t) - p(x_i, t|x_j, t)}{\Delta t}, \tag{5.25}$$

$p(x_i, t|x_j, t) = \delta(x_i - x_j)$. Markov processes describe a variety of physical phenomena. In practice, a process may be specified in terms of its transition rates. If it is a Markov process, then its probability density can be shown to evolve according to the *master equation* [66]

$$\frac{\partial}{\partial t}p(xt) = \int \mathrm{d}y\Big[w(x|y)p(yt) - w(y|x)p(xt)\Big]. \tag{5.26}$$

This may be interpreted as a probability balancing equation. For, the first term on the right hand side of (5.26) represents the rate at which probability of assuming a value x is gained at the cost of the probabilities of the other values whereas the second term gives the rate at which the probabilities of the other values gain at the cost of the probability of the value x. The differential equation form of this equation is the *Kramers-Moyal expansion* [65]

$$\dot{p}(xt) = \hat{L}\,(x,t)\,p(xt), \tag{5.27}$$

$$\hat{L}\,(x,t) = \sum_{m=1}^{\infty}\frac{1}{m!}\left(-\frac{\partial}{\partial x}\right)^m D_m(xt), \tag{5.28}$$

$$D_m = Lt_{\Delta t\to 0}\frac{\langle\{\xi(t+\Delta t) - \xi(t)\}^m\rangle}{\Delta t}\bigg|_{\xi(t)=x}. \tag{5.29}$$

The equation (5.27) determines the evolution of the probability in terms of the moments of the process. Its formal solution yields

$$p(x, t|x_0, t_0) = \overleftarrow{T}\exp\left[\int_{t_0}^{t}\mathrm{d}\tau\hat{L}\,(x,\tau)\right]\delta(x - x_0). \tag{5.30}$$

Verify by combining this with (5.25) that the rate of transition may be expressed as

$$w(x|x') = \hat{L}\,(x,t)\,\delta\,(x - x')\,. \tag{5.31}$$

Now, the quantities of practical interest are the averages of functions of x. The equation for the average of a function $f(x)$ can be found by multiplying (5.27) by $f(x)$ and integrating with respect to x to get

$$\langle\dot{f}(x)\rangle = \int \mathrm{d}x\; p(x)\hat{L}^{\dagger}\,(x,t)\;\; f(x), \tag{5.32}$$

where $\hat{L}^{\dagger}(x)$ is the adjoint of $\hat{L}(x)$ defined by the relation

$$\int \mathrm{d}x\; g(x)\hat{L}(x)h(x) = \int \mathrm{d}x\; h(x)\hat{L}^{\dagger}(x)g(x), \tag{5.33}$$

with $h(x)$ and $g(x)$ obeying appropriate boundary conditions. Verify by repeated partial integration that if the functions in question and their derivatives vanish at the boundaries then

$$\int \mathrm{d}x\ g(x)\frac{\mathrm{d}^m}{\mathrm{d}x^m}h(x) = (-)^m \int \mathrm{d}x\ h(x)\frac{\mathrm{d}^m}{\mathrm{d}x^m}g(x). \tag{5.34}$$

On comparing this with (5.33) we see that

$$\left(\frac{\mathrm{d}^m}{\mathrm{d}x^m}\right)^\dagger = (-)^m \frac{\mathrm{d}^m}{\mathrm{d}x^m}. \tag{5.35}$$

In the foregoing, we assumed that the random function assumes continuously distributed real values. We will have occasions to deal with random functions which assume only integral values. The master equation (5.26) then reads

$$\frac{\partial}{\partial t}p(m,t) = \sum_n \left[w(m|n)p(n,t) - w(n|m)p(m,t)\right], \tag{5.36}$$

the $p(m,t)$ being the probability that the value of the random function at time t is m and $w(n|m)$ is the rate with which its value m makes a transition to the value n.

The equation for single time probability of a Markov process enables us to determine its multi-time correlations as well. This property is underlined by the *regression theorem*. It states that if $\xi_1(t), \ldots, \xi_N(t)$ are N random functions of time described by a Markov process such that their averages evolve according to

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \xi_i(t)\rangle = \sum_{j=1}^{N} a_{ij}\langle \xi_j(t)\rangle, \tag{5.37}$$

where a_{ij} are independent of time then the evolution of their two-time correlations is governed by

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \xi_i(t)\xi_k(t_0)\rangle = \sum_{j=1}^{N} a_{ij}\langle \xi_j(t)\xi_k(t_0)\rangle, \tag{5.38}$$

This equation is the same as (5.37) for single time average. See [65, 68] for its proof.

5.3 Detailed Balance

If $\hat{L}(x)$ is time-independent then (5.27) may be solved by the method of eigenfunction expansion discussed in Chap. 10. Let zero be a non-degenerate eigenvalue of $\hat{L}(x)$ and let the real part of all its other eigenvalues be negative. Consequently, as $t \to \infty$, $p(x\dot{t}) \to p_{\mathrm{ss}}(x)$ where $p_{\mathrm{ss}}(x)$, called the *steady state* distribution function, is the eigenfunction corresponding to the eigenvalue zero:

$$\dot{p}_{\mathrm{ss}}(x) = \hat{L}(x)p_{\mathrm{ss}}(x) = 0. \tag{5.39}$$

The solution of the steady state equation is greatly facilitated if the process obeys the condition of detailed balance discussed next.

Consider a stationary Markov process described by N random functions $\xi_1(t), \ldots, \xi_N(t)$. Let $w(\{x_i'\}|\{x_i\})$ be the rate of transition from the values $\{x_i\}$ to $\{x_i'\}$ and let $p_{\rm ss}(\{x_i\})$ be the corresponding steady state. Under the operation of time-reversal $t \to -t$, let $x_i \to \epsilon_i x$, $\{\mu_i\} \to \{\epsilon_i \mu_i\}$ where $\{\mu_i\}$ denotes a set of external variables and $\epsilon_i = \pm 1$. The process is said to be in *detailed balance* if

$$p_{\rm ss}(\{x_i\}, \{\mu_i\}) = p_{\rm ss}(\{\epsilon_i x_i\}, \{\epsilon_i \mu_i\}), \tag{5.40a}$$

$$w(\{\epsilon_i x_i'\}|\{\epsilon_i x_i\}) p_{\rm ss}(\{x_i\}, \{\mu_i\}) = w(\{x_i\}|\{x_i'\}) p_{\rm ss}(\{x_i'\}, \{\mu_i\}). \tag{5.40b}$$

The first of the conditions above is the statement of invariance of the steady state distribution under the time reversal. The second condition states that the number of transition from a state described by the values $\{x_i\}$ to another state described by the values $\{x_i'\}$ is the same as the number of transition from $\{x_i'\}$ to $\{x_i\}$ in the reverse direction of time. Using (5.31) to express $w(\{y\}|\{z\})$ in terms of $\hat{L}(\{y\})$, and after a little algebra, (5.40b) may be reduced to the form [66]

$$\begin{aligned} & p_{\rm ss}\left(\{x_i\}, \{\mu_i\}\right) \hat{L}^{\dagger}\left(\{\epsilon_i x_i\}, \{\epsilon \mu_i\}\right) F\left(\{x_i\}\right) \\ & = \hat{L}\left(\{x_i\}, \{\mu_i\}\right) p_{\rm ss}\left(\{x_i\}, \{\mu_i\}\right) F\left(\{x_i\}\right), \end{aligned} \tag{5.41}$$

the F being an arbitrary function. This is the detailed balance condition in terms of the operator governing the evolution of the probability density of a Markov process. We demonstrate the usefulness of this condition in the next section for finding $p_{\rm ss}$.

The detailed balance condition for the case of discretely varying random variable described by the master equation (5.36) reads

$$w(m|n) p_n^{\rm ss} = w(n|m) p_m^{\rm ss}. \tag{5.42}$$

This states that the number of transitions in unit time from the value n to the value m in the steady state is the same as that from m to n.

5.4 Liouville and Fokker–Planck Equations

An advantage of working with the Kramers–Moyal expansion (5.27) lies in the possibility of terminating it at some finite step. However, a theorem due to Pawula (see [66]) asserts that the solution of (5.27) is everywhere non-negative if it is truncated at the first or at the second term but not if it is truncated at any other finite step. Since the admissible solutions of (5.27) should be non-negative, it follows that it can be terminated up to the second step or else all its terms should be retained. This theorem holds also for a multivariate system.

5.4.1 Liouville Equation

The expansion (5.28), generalized to N variables and truncated at the first step, reduces (5.27) to the *Liouville equation*

$$\dot{p}(\{x_i\},t) = -\sum_i \frac{\partial}{\partial x_i} A_i(\{x_i\}) p(\{x_i\},t), \tag{5.43}$$

$$A_i(\{x_i\}) = Lt_{\Delta t\to 0} \frac{\langle\{\xi_i(t+\Delta t) - \xi_i(t)\}\rangle}{\Delta t}\Big|_{\xi_i(t)=x_i}. \tag{5.44}$$

To find the solution of (5.43), consider the equation

$$\dot{z}_i(t) = A_i(\{z_i(t)\}), \tag{5.45}$$

with $z_i(t_0) = y_i$ as the initial condition. Let $z_i(\{y_i\},t)$ be the corresponding solution of (5.45). Let $p(\{x_i\},t_0) = \delta(\{x_i - y_i\})$. Verify by direct substitution that (5.43) is solved by

$$p(\{x_i\},t|\{y_i\},t_0) = \delta(\{x_i - z_i(\{y_i\},t)\}). \tag{5.46}$$

This means that, for given initial conditions, the system traverses a definite path determined by the solution of (5.45). For details, see [65].

5.4.2 The Fokker–Planck Equation

The multivariate form of (5.28) terminated at the second term reduces (5.27) to the *Fokker–Planck equation*

$$\dot{p}(\{x_k\},t) = \hat{L}_{\rm FP}(\{x_k\}) p(\{x_k\},t), \tag{5.47}$$

where the Fokker–Planck operator is given by

$$\hat{L}_{\rm FP}(\{x_k\}) = \sum_i \Big[-\frac{\partial}{\partial x_i} A_i(\{x_k\},t) + \frac{1}{2}\sum_{j=1}^{N} \frac{\partial^2}{\partial x_i \partial x_j} D_{ij}(\{x_k\},t)\Big]. \tag{5.48}$$

The column of the elements $\{A_i(\{x_i\})\}$, defined in (5.44), constitutes the *drift vector* whereas

$$D_{ij}(\{x_k\}) = Lt_{\Delta t\to 0} \frac{\langle(\xi_i(t+\Delta t) - \xi_i(t))(\xi_j(t+\Delta t) - \xi_j(t))\rangle}{\Delta t}\Big|_{\xi_k(t)=x_k} \tag{5.49}$$

constitute the *diffusion matrix*. The diffusion matrix is positive. The Fokker–Planck equation is a continuity equation for the flow of the probability as is revealed by writing it in the form

$$\dot{p}(\{x_i\},t) + \sum_{i=1}^{N} \frac{\partial J_i}{\partial x_i} = 0 \tag{5.50}$$

where

$$J_i = \left[A_i(\{x_i\},t) - \frac{1}{2}\sum_{j=1}^{N}\frac{\partial}{\partial x_j}D_{ij}(\{x\},t)\right]p(\{x\},t). \tag{5.51}$$

is the *probability current.*

Now, if $\hat{L}_{FP}(\{x_i\})$ is time-independent, then the steady state $p_{ss}(\{x_i\})$ of the Fokker–Planck equation is obtained by solving

$$\sum_i \frac{\partial J_i}{\partial x_i} = 0. \tag{5.52}$$

For a univariate process, this implies $J =$ constant, i.e.

$$\left[A - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}D(x)\right]p_{ss}(x) = C, \tag{5.53}$$

where C is a constant to be determined by the boundary conditions. If J is assumed to vanish at the boundaries then $C = 0$. In this case

$$p_{ss}(x) = \frac{N}{D(x)}\exp(\Phi(x)), \qquad N^{-1} = \int \mathrm{d}x \frac{1}{D(x)}\exp(\Phi(x)), \tag{5.54}$$

the $\Phi(x)$ being the *potential function* defined by

$$\Phi(x) = 2\int \mathrm{d}x\ \frac{A(x)}{D(x)}. \tag{5.55}$$

(5.54) shows that p_{ss} is peaked at the maxima of $\Phi(x)$.

However, if the process is multivariate then (5.52) does not necessarily imply $J_i =$ constant for each i. Solving (5.52) is generally a formidable task. This task is simplified considerably if the process satisfies the condition (5.41) of the detailed balance. In order to see that, substitute the Fokker–Planck operator $\hat{L}_{FP}$ of (5.48) for $\hat{L}$ in (5.41). Note that, by virtue of its definition, $D(\{\epsilon_i x_i, \epsilon_j x_j\}) = \epsilon_i\epsilon_j D(\{x_i, x_j\})$. Now, equate to zero separately the coefficients of $F(\{x_i\})$ and of $\mathrm{d}F(\{x_i\})/\mathrm{d}x_i$ to show that $p_{ss}(\{x_i\})$ solves

$$\sum_i \frac{\partial}{\partial x_i}\Big[A_i(\{x_i\}) - \epsilon_i A_i(\{\epsilon_i x_i\})\Big]p_{ss}(\{x_i\}) = 0,$$

$$\Big[A_i(\{x_i\}) + \epsilon_i A_i(\{\epsilon_i x_i\}) - \sum_j \frac{\partial}{\partial x_j}D_{ij}\Big]p_{ss}(\{x_i\}) = 0. \tag{5.56}$$

The task of finding p_{ss} for a detailed-balanced process is thus reduced to solving first-order equations. The equation to be solved in the absence of the detailed balance is a multivariate second-order partial differential equation.

We have outlined above the way of finding the probability density of a process described in terms of its moments. In many practical situations, a system is described by equations of evolution of its dynamical variables. In the next section we discuss how to extract the stochastic properties of a system from its dynamical equations.

5.5 Stochastic Differential Equations

An equation of motion of a dynamical variable subject to the influence of a random force is named a *stochastic differential equation* (s.d.e.). We consider the s.d.e. written in the form

$$\dot{\xi}_i = f_i(\{\xi_i\}, t) + \sum_j g_{ij}(\{\xi_i\}, t)\eta_j(t), \tag{5.57}$$

where the first term on the right hand side is a deterministic function of time and of $\{\xi_i(t)\}$ whereas $\eta_i(t)$ in the second term are random functions of time. Given the probability distributions of $\eta_i(t)$, our aim is to derive those for $\{\xi_i\}$.

Solution of a stochastic differential equation involves integration over random functions of time. For details of the theory of integration of stochastic differential equations, see [65]. Of several ways of defining integration involving random functions, it turns out that the definition of Stratonovich enables one to use the rules of the theory of the calculus of ordinary functions [65]. In what follows, we treat the stochastic integration in the same way as the integration of ordinary functions. In other words, we implicitly follow the approach of Stratonovich.

It is known that if $\eta_i(t)$ are *delta correlated*, i.e. if $\langle\langle \eta_1(t_1)\cdots\eta_n(t_n)\rangle\rangle = D(t_1)\delta(t_1-t_2)\cdots\delta(t_1-t_n)$ then $\xi(t)$ is Markovian [64]. The equation governing the evolution of a probability density of the process described by (5.57) can be derived analytically in general if $\{\eta_i(t)\}$ is a stationary delta-correlated Gaussian process characterized by

$$\begin{aligned} &\langle\eta_i(t)\rangle = 0, \quad \langle\eta_i(t)\eta_j(t')\rangle = D_i\delta_{ij}\delta(t-t'), \\ &\langle\langle\eta_i(t_1)\cdots\eta_i(t_n)\rangle\rangle = 0, \quad n \geq 3. \end{aligned} \tag{5.58}$$

Note that the spectral density $S(\omega)$, defined in (6.91), is independent of ω if the process is delta-correlated. Hence a delta-correlated process is also called a *white noise*. The processes for which $S(\omega)$ exhibits dependence on ω are said to constitute *coloured noise*. The process described by (5.58) is the *Gaussian white noise*.

Now, the definition (5.1) of single time probability density of a univariate process generalized to n random functions $\xi_1(t), \ldots, \xi_N(t)$ reads

$$p(\{x_i\}, t) = \langle\rho\rangle, \quad \rho = \prod_{i=1}^{N} \delta\left(\xi_i(t) - x_i\right). \tag{5.59}$$

Differentiate ρ with respect to time and substitute (5.57) for $\dot{\xi}_i$. Use next the property (A.7) of the delta-function to show that

$$\dot{\rho} = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big[f_i(\{x_i\}, t) + \sum_{j=1}^{N} g_{ij}(\{x_i\}, t)\eta_j(t)\Big]\rho. \tag{5.60}$$

Derivation of the equation for the probability density $p(\{x_i\},t)$ is facilitated by the following theorem:

Theorem 5.1: Let $\boldsymbol{y}(\{x\},(t))$ *be a vector function of variables* $(\{x\})$ *obeying*

$$\dot{\boldsymbol{y}}(\{x\},t) = \Big[\hat{A} + \sum_{k=1} \hat{B}_k \eta_k(t)\Big]\boldsymbol{y}(\{x\},t), \tag{5.61}$$

where $\hat{A}(\{x\})$ *and* $\hat{B}_k(\{x\})$ *are matrices whose elements are differential operators of* $\{x\}$. *If* $\{\eta_k(t)\}$ *is a Gaussian white noise characterized by (5.58) then* $\langle \boldsymbol{y}(\{x\},t)\rangle$ *obeys*

$$\langle \dot{\boldsymbol{y}}(\{x\},t)\rangle = \Big[\hat{A} + \frac{1}{2}\sum_{k=1} D_k \hat{B}_k^2\Big]\langle \boldsymbol{y}(\{x\},t)\rangle \tag{5.62}$$

where the average is over the distribution of $\{\eta_k(t)\}$. We refer to [69] for its proof. On applying this theorem to (5.60) and on recalling from (5.59) that $\langle\rho\rangle = p(\{x_i\},t)$, we find that $p(\{x_i\},t)$ obeys the Fokker–Planck equation

$$\dot{p}(\{x_i\},t) = \Big[-\sum_i \frac{\partial}{\partial x_i} f_i(\{x_i\},t) + \frac{1}{2}\sum_{l,j,k=1}^{N} D_k \frac{\partial}{\partial x_l} g_{lk}(\{x_i\},t)\frac{\partial}{\partial x_j} g_{jk}(\{x_i\},t)\Big] p(\{x_i\},t). \tag{5.63}$$

If $p(\{x_i\},t_0) = \delta(x_1 - y_1)\cdots\delta(x_N - y_N)$ then the solution of (5.63) gives the transition probability $p(\{x_i\},t|\{y_i\},t_0)$. Since delta-correlation of $\{\eta_k(t)\}$ ensures that $\{\xi_i(t)\}$ is Markovian [64], the transition and single time probabilities determine all its multi-time correlation functions.

In the next two sections we discuss some analytically solvable cases of (5.57). To that end, it is useful to classify the equations according to how the system is coupled to the random influence $\{\eta_k(t)\}$. If it is so coupled that $\{g_{ij}\}$ are independent of $\{\xi_i(t)\}$ then the noise induced by $\{\eta_k(t)\}$ is called *additive*. Else it is termed a *multiplicative noise*. There are, however, situations which permit transformation between the two types of noises [66, 67].

5.6 Linear Equations with Additive Noise

In this section we solve the linear s.d.e.

$$\dot{\xi}(t) = f(t)\xi(t) + g(t)\eta(t). \tag{5.64}$$

If $\eta(t)$ is a Gaussian white noise then $\xi(t)$ is said to describe a *Wiener process* if $f(t) = 0$, and an *Ornstein–Uhlenbeck process* (OU) if $f(t) \neq 0$.

The formal solution of (5.64) is

$$\xi(t) = a(t,t_0)x_0 + \int_{t_0}^{t} \mathrm{d}\tau\; b(t,\tau)\eta(\tau), \qquad x_0 = \xi(t_0), \tag{5.65}$$

$$a(t,t_0) = \exp\left(\int_{t_0}^{t} \mathrm{d}\tau f(\tau)\right), \qquad b(t,\tau) = g(\tau)a(t,\tau). \tag{5.66}$$

We insert (5.65) in the definition (5.9) of the characteristic function to obtain

$$\begin{aligned} C_1(u) &= \langle \exp(\mathrm{i}u\xi(t)) \rangle \\ &= \exp\left[\mathrm{i}ua(t,t_0)x_0\right] \left\langle \exp\left[\mathrm{i}u \int_{t_0}^{t} \mathrm{d}\tau\ b(t,\tau)\eta(\tau)\right] \right\rangle. \end{aligned} \tag{5.67}$$

Apply (5.13) to express this equation in the form

$$C_1(u) = \exp\left[\mathrm{i}ua(t,t_0)x_0\right] \exp\left[\sum_{m=1}^{\infty} \frac{(\mathrm{i}u)^m}{m!} K_m(t,t_0)\right], \tag{5.68}$$

$$\begin{aligned} &K_m(t,t_0) \\ &= \int_{t_0}^{t} \mathrm{d}\tau_m \cdots \int_{t_0}^{t} \mathrm{d}\tau_1 \langle\langle \eta(\tau_1) \cdots \eta(\tau_m) \rangle\rangle b(t,\tau_1) \cdots b(t,\tau_m). \end{aligned} \tag{5.69}$$

Since $C_1(u)$ constructed above is under the condition that $\xi(t_0) = x_0$, its substitution in (5.8) yields the conditional probability $p(x,t|x_0,t_0)$. The integral in (5.8) can be evaluated analytically exactly if $C_1(u)$ is a Gaussian, i.e. if $K_m(t) = 0$ for $m > 2$. This, in turn, requires that $\eta(t)$ be Gaussian. By assuming that to be the case and with $\langle \eta(t) \rangle = 0$ we get

$$p(x,t|x_0,t_0) = \sqrt{\frac{1}{2\pi\ K_2(t,t_0)}} \exp\left[-\frac{\Big(x - a(t,t_0)x_0\Big)^2}{2K_2(t,t_0)}\right]. \tag{5.70}$$

Verify by direct substitution that $p(x,t|x_0,t_0)$ obeys the Chapman-Kolmogorov equation (5.24) for a Markov process only if if $\eta(t)$ is delta-correlated. If $\eta(t)$ is not delta-correlated then we need to construct multi time probability densities for a complete characterization of the process described by $\xi(t)$. The multi time probabilities may be derived in a manner similar to the one followed above for deriving the single time probability.

Let $\eta(t)$ be a Gaussian white noise characterized by (5.58). Also, let $f(t) = 0$ so that $\xi(t)$ describes a Wiener process. If, in addition, $g = 1$ then $a(t,t_0) = 1$, $b(t,\tau) = 1$, $K_2(t,t_0) = D(t - t_0)$. The equation (5.70) then assumes the form

$$p(xt|x_0t_0) = \sqrt{\frac{1}{2\pi\ D(t-t_0)}} \exp\left[-\frac{\Big(x - x_0\Big)^2}{2D(t-t_0)}\right]. \tag{5.71}$$

Next, let $f(t) = -\gamma$ $(\gamma > 0)$, $g(t) = 1$ so that $\xi(t)$ describes an OU process. On inserting these values in (5.66) and (5.69) we find that

$$\begin{aligned} a(t,t_0) &= \exp\{-\gamma(t-t_0)\}, \qquad b(t,\tau) = a(t,\tau) \\ K_2(t,t_0) &= \frac{D}{2\gamma}\Big\{1 - \exp(-2\gamma(t-t_0))\Big\}. \end{aligned} \tag{5.72}$$

The expression (5.70) then assumes the form

$$p(x,t|x_0,t_0) = \sqrt{\frac{\gamma}{\pi D\Big(1-\exp(-2\gamma(t-t_0))\Big)}} \times \exp\left[-\frac{\gamma}{D}\frac{\Big\{x-\exp(-\gamma(t-t_0)x_0)\Big\}^2}{1-\exp(-2\gamma(t-t_0))}\right]. \tag{5.73}$$

In the limit $t \to \infty$ it yields

$$p(xt|x_0t_0) \to \sqrt{\frac{\gamma}{\pi D}}\exp\Big(-\frac{\gamma}{D}x^2\Big), \quad (\gamma > 0). \tag{5.74}$$

The asymptotic distribution is independent of the initial distribution.

It is straightforward to verify that (5.73) obeys the Fokker–Planck equation

$$\frac{\partial p(x,t|x_0,t_0)}{\partial t} = \left[\gamma\frac{\partial}{\partial x} + \frac{D}{2}\frac{\partial^2}{\partial x^2}\right]p(x,t|x_0,t_0). \tag{5.75}$$

This is the same as (5.63) specialized to the present situation.

Next, use (5.65) to show that

$$\langle\xi(t_1)\xi(t_2)\rangle = \exp(-\gamma(t_1+t_2)) \times \Big[x_0^2 + D\int_0^{t_1}\int_0^{t_2} \mathrm{d}\tau_1\mathrm{d}\tau_2\exp(\gamma(\tau_1+\tau_2))\delta(\tau_1-\tau_2). \tag{5.76}$$

The δ-function reduces the double integral above to single integral in which the upper limit is the smaller of t_1 and t_2. Assume also that $\gamma t_1, \gamma t_2 \gg 1$. The expression (5.76) then reduces to

$$\langle\xi(t_1)\xi(t_2)\rangle = \frac{D}{2\gamma}\exp\{-\gamma|t_1-t_2|\}. \tag{5.77}$$

This is the two-time correlation of an OU process in the long time limit.

Generalization of these results to multivariate systems is straightforward.

5.7 Linear Equations with Multiplicative Noise

In this section we discuss the problem of solving

$$\dot{\boldsymbol{\xi}} = \Big[\hat{F}(t) + \hat{G}(t)\eta(t)\Big]\boldsymbol{\xi}. \tag{5.78}$$

Here $\boldsymbol{\xi}$ is a column vector constituted by the elements $(\xi_1(t),\ldots,\xi_N(t))$; $\hat{F}(t)$ and $\hat{G}(t)$ are the $N\times N$ matrices independent of $\{\xi(t)\}$. We discuss separately the univariate and the multivariate cases.

5.7.1 Univariate Linear Multiplicative Stochastic Differential Equations

The univariate form of (5.78) reads

$$\dot{\xi} = \Big[f(t) + \eta(t)g(t)\Big]\xi. \tag{5.79}$$

Its formal solution is given by ($x_0 = \xi(0)$)

$$\xi(t) = x_0 a(t) \exp\left[\int_0^t \mathrm{d}\tau \eta(\tau) g(\tau)\right], \quad a(t) = \exp\left[\int_0^t f(\tau)\mathrm{d}\tau\right]. \tag{5.80}$$

On raising this to power m and on invoking (5.13) we obtain

$$\begin{aligned}\langle \xi^m(t)\rangle &= x_0^m a^m(t) \exp\left[\sum_{n=1}^{\infty} \frac{1}{n!} K_n(t)\right], \\ K_n(t) &= m^n \int_{t_0}^t \mathrm{d}\tau_1 \cdots \int_{t_0}^t \mathrm{d}\tau_n \langle\langle \eta(\tau_1)\cdots\eta(\tau_n)\rangle\rangle g(\tau_1)\cdots g(\tau_n).\end{aligned} \tag{5.81}$$

If $\eta(t)$ is a Gaussian with zero mean then

$$\langle \xi^m(t)\rangle = x_0^m a^m(t) \exp\Big[\frac{m^2}{2} K_2(t)\Big]. \tag{5.82}$$

Substitute this in (5.9) to obtain

$$C_1(u) = \sum_{m=0}^{\infty} \frac{1}{m!} \Big[\mathrm{i}u x_0 a(t)\Big]^m \exp\Big[\frac{m^2}{2} K_2(t)\Big]. \tag{5.83}$$

Now, rewrite the exponential in (5.83) recalling (A.20) (with $\beta = m, \alpha = 1/2K_2(t)$) so that

$$\begin{aligned}C_1(u) = \sqrt{\frac{1}{2\pi K_2(t)}} \int_{-\infty}^{\infty} \mathrm{d}y \Big[\sum_{m=0}^{\infty} \frac{1}{m!} \Big[\mathrm{i}u x_0 a(t) \exp(y)\Big]^m \\ \times \exp(-y^2/2K_2(t))\Big].\end{aligned} \tag{5.84}$$

The summation over m can now be carried. Substitute the resulting expression in (5.8). The integration over u is a delta function in $x - x_0 a(t)\exp(y)$. This enables us to integrate over y and obtain the probability distribution as

$$p(x,t|x_0,0) = \sqrt{\frac{1}{2\pi K_2(t) x^2}} \exp\Big[-\frac{1}{2K_2(t)} \ln^2\Big\{\frac{x}{x_0 a(t)}\Big\}\Big]. \tag{5.85}$$

For an application of this result in laser physics, see [68].

5.7.2 Multivariate Linear Multiplicative Stochastic Differential Equations

Now we examine (5.78) when ξ is an N-dimensional column vector. We assume $\hat{F}$ and $\hat{G}$ to be independent of time. Define

$$\boldsymbol{\xi}'(t) = \exp(-t\hat{F})\boldsymbol{\xi}(t) \tag{5.86}$$

so that

$$\dot{\boldsymbol{\xi}}'(t) = \eta(t)\hat{\tilde{G}}(t)\boldsymbol{\xi}'(t), \quad \hat{\tilde{G}} = \exp(-t\hat{F})\hat{G}\exp(t\hat{F}). \tag{5.87}$$

Its formal solution, averaged over the distribution of $\eta(t)$, is

$$\langle\boldsymbol{\xi}'(t)\rangle = \left\langle \overleftarrow{T} \exp\left[\int_{t_0}^{t} \mathrm{d}\tau\eta(\tau)\hat{\tilde{G}}(\tau)\right]\right\rangle \boldsymbol{\xi}(0). \tag{5.88}$$

Recall from Sect. 1.1 that if $\left[\hat{\tilde{G}}(t_1),\ \hat{\tilde{G}}(t_2)\right] = 0$ for all t_1, t_2 then the time-ordered exponential integral is simply the exponential of the integral. This condition will be satisfied if $\left[\hat{F},\ \hat{G}\right] = 0$. Assuming this to be the case, we invoke (5.13) to perform the cumulant expansion of the average of the exponential of the integral and transform back to $\boldsymbol{\xi}$ to get

$$\begin{aligned}\langle\boldsymbol{\xi}(t)\rangle &= \exp(t\hat{F}) \\ &\times \exp\left[\sum_{n=1}^{\infty}\frac{1}{n!}\int_{t_0}^{t}\mathrm{d}\tau_1\cdots\int_{t_0}^{t}\mathrm{d}\tau_n\langle\langle\eta(\tau_1)\cdots\eta(\tau_n)\rangle\rangle\hat{G}^n\right]\boldsymbol{\xi}(0).\end{aligned} \tag{5.89}$$

If $\left[\hat{F},\ \hat{G}\right] \neq 0$ then (5.88) can be evaluated if $\eta(t)$ is a Gaussian white noise. For such an $\eta(t)$, we can apply theorem 5.1 to (5.78) to get

$$\langle\dot{\boldsymbol{\xi}}\rangle = \left[\hat{F}(t) + \frac{D}{2}\hat{G}^2(t)\right]\langle\boldsymbol{\xi}\rangle. \tag{5.90}$$

The problem of solving (5.78) for a multivariate system driven by coloured $\eta(t)$ is generally analytically formidable one. A case of coloured noise of common interest is when $\eta(t)$ is a Gaussian with zero mean and the two-time correlation given by

$$\langle\eta(t_1)\eta(t_2)\rangle = \frac{D\gamma}{2}\exp(-\gamma|t_1 - t_2|). \tag{5.91}$$

The form of the coefficient of the exponential above ensures that, in the limit $\gamma \to \infty$ it reduces, by virtue of (A.8), to $D\delta(t_1 - t_2)$. The correlation in (5.91) is of the form (5.77) which arises in the long-time limit of the OU process. We, therefore, assume $\eta(t)$ to be an OU process governed by

$$\dot{\eta}(t) = -\gamma\eta + \alpha(t), \tag{5.92}$$

where $\alpha(t)$ is Gaussian white noise with zero mean and the two-time correlation given by

$$\langle \alpha(t)\alpha(t')\rangle = D\gamma^2\delta(t-t'). \tag{5.93}$$

The problem now boils down to solving the coupled equations: equation (5.78) for $\boldsymbol{\xi}(t)$ and (5.93) for $\eta(t)$. The noise in these equations is a Gaussian white noise contributed by $\alpha(t)$. These equations may be combined to resemble (5.57) whose probability distribution obeys (5.63) if the noise is Gaussian white noise. To that end, we let $\xi_{N+1} = \eta(t)$ so that the set of equations

$$\dot{\xi}_i = \sum_{j=1}^{N} \Big[F_{ij}(t) + G_{ij}(t)\xi_{N+1}\Big](1-\delta_{iN+1})\xi_j - \gamma\delta_{iN+1}\xi_{N+1} + \alpha(t)\delta_{iN+1}, \tag{5.94}$$

$i = 1, 2, \ldots, N+1$, is equivalent to (5.78) and (5.92). If $x_1, \ldots, x_n$ and y stand for realizable values of $\xi_1(t), \ldots, \xi_n(t)$ and ξ_{N+1} then the probability density $p(\boldsymbol{x}, y, t)$ obeys

$$\frac{\partial}{\partial t}p(\mathbf{x}, y, t) = \sum_{i,j=1}^{N}\Big[-\frac{\partial}{\partial x_i}(F_{ij} + G_{ij}y)x_j\Big]p(\mathbf{x}, y, t) + \Big[\gamma\frac{\partial}{\partial y}y + \frac{D\gamma^2}{2}\frac{\partial^2}{\partial y^2}\Big]p(\mathbf{x}, y, t). \tag{5.95}$$

Our interest is in evaluating the averages

$$\langle \xi_k\rangle = \int x_k\, p(\mathbf{x}, y, t)\mathrm{d}x_1\cdots\mathrm{d}x_N\mathrm{d}y \equiv \int z_k\mathrm{d}y. \tag{5.96}$$

where

$$z_k = \int x_k p(\mathbf{x}, y, t)\mathrm{d}x_1\cdots\mathrm{d}x_N. \tag{5.97}$$

On multiplying (5.95) by x_k and by integrating over $x_1, \ldots, x_n$ we find that z_k obeys the equation

$$\frac{\partial}{\partial t}z_k = \sum_{j=1}^{N}\Big[(F_{kj} + G_{kj}y)z_j\Big] + \Big[\gamma\frac{\partial}{\partial y} + \frac{D\gamma^2}{2}\frac{\partial^2}{\partial y^2}\Big]z_k. \tag{5.98}$$

This equation is solvable exactly for a univariate system. We have anyway already solved the univariate problem in Sect. 5.7.1 for most general noise. For the details of the methods of approximate analytical and numerical solution of (5.98), see [66]. In the context of quantum optics, this finds application in the study of the dynamics of a two-level atom in a fluctuating field in which noise is characterized by (5.91) [70].

5.8 The Poisson Process

We have so far considered continuously varying random functions of time. In this section we consider a discrete random process, the so called *Poisson*

Process. This process is described by a random variable $\xi(t)$ whose value increases randomly by one as a function of time without correlation between any two values. The numbers $m = 0, 1, \ldots$ are the realizations of the values of $\xi(t)$. Let $p(m,t)$ be the probability that $\xi(t) = m$ at time t. Let R be the rate of increase of $\xi(t)$ so that $w(n|m) = R\delta_{n,m+1}$. The master equation (5.36) for $p(m,t)$ then reduces to

$$\dot{p}(m,t) = R\Big[p(m-1,t) - p(m,t)\Big]. \tag{5.99}$$

If $p(m,0) = \delta_{m0}$ then this is evidently solved by

$$p(m,t) = \frac{(Rt)^m}{m!}\exp(-Rt). \tag{5.100}$$

By using the defining relation

$$\langle f(\xi(t))\rangle = \sum_{m=0}^{\infty} f(m)p(m,t) \tag{5.101}$$

for an average, verify that

$$C_1(u) = \langle \exp(\mathrm{i}u\xi(t))\rangle = \exp\Big[Rt\Big(\exp(\mathrm{i}u) - 1\Big)\Big]. \tag{5.102}$$

Compare this with the cumulant expansion (5.12) of $C_1(u)$ and verify that

$$\langle\langle \xi^N(t)\rangle\rangle = Rt. \tag{5.103}$$

This states that all single time cumulants of the Poisson process are equal. Recall from (5.14) that $\langle\langle \xi(t)\rangle\rangle$ is the mean of $\xi(t)$. Hence all single time cumulants of a Poisson process are equal to its mean.

5.9 Stochastic Differential Equation Driven by Random Telegraph Noise

In this section we outline a method of solving

$$\dot{\boldsymbol{\xi}}(t) = \hat{M}(\eta(t))\boldsymbol{\xi}(t) \tag{5.104}$$

where the matrix $\hat{M}(\eta(t))$ is a function of the *telegraph noise* $\eta(t)$. The function $\eta(t)$ is such that (i) it jumps between discretely spaced values $a, b, \cdots$ at a rate R and remains constant between the jumps, (ii) the jumps occur randomly without any correlation, (iii) the process is stationary. Now, let us denote by $\boldsymbol{x}(\mu,t)$ the value of $\boldsymbol{\xi}(t)$ at time t in a realization in which the value of $\eta(t)$ is μ. The value of $\boldsymbol{\xi}(t)$ in different realizations in which $\eta(t) = \mu$ need not be same. This is because evolution to time t depends on the values of $\eta(t)$ at earlier times which are random. Let $\langle \boldsymbol{x}(\mu,t)\rangle$ denote the average of $\boldsymbol{\xi}(t)$ under the condition that $\eta(t) = \mu$. Clearly

$$\langle \boldsymbol{\xi}(t)\rangle = \sum_{\mu=a,b\ldots} \langle \boldsymbol{x}(\mu,t)\rangle \tag{5.105}$$

Let us consider a realization in which $\eta(t) = \mu$. We assume that $\eta(t)$ does not change during the evolution of $\xi(t)$ till time $t + \mathrm{d}t$. The solution of (5.104) gives $\boldsymbol{\xi}(t + \mathrm{d}t) = (1 + \hat{M}(\mu)\mathrm{d}t)\boldsymbol{x}(\mu, t)$. The $\eta(t)$ may or may not change its value at $t + \mathrm{d}t$. Let $\eta(t + \mathrm{d}t) = \nu$ so that the value of $\boldsymbol{\xi}(t + \mathrm{d}t)$ is denoted by $\boldsymbol{x}(\nu, \mathrm{d}t + t)$. The probable value of $\eta(t + \mathrm{d}t)$ is found by introducing the probability that $\eta(t)$ jumps from its value μ to ν in time $\mathrm{d}t$. It is given by $w(\nu|\mu)(\mathrm{d}t) = Rp(\nu|\mu)\mathrm{d}t$ $(\mu \neq \nu)$, R being the rate of transition. The probability of transition $p(\nu|\mu)$ obeys

$$\sum_{\mu=a,b,\ldots} p(\nu|\mu) = 1. \tag{5.106}$$

Hence, the probability that the value of the variable does not change in time $\mathrm{d}t$ is $1 - R\mathrm{d}t$. It then follows that

$$\begin{aligned}\langle \boldsymbol{x}(\nu, t + \mathrm{d}t)\rangle = {} & (1 - R\mathrm{d}t)(1 + \hat{M}(\nu)\mathrm{d}t)\langle \boldsymbol{x}(\nu, t)\rangle \\ & + R\sum_{\mu} p(\nu|\mu)\mathrm{d}t \left(1 + \hat{M}(\mu)\mathrm{d}t\right) \langle \boldsymbol{x}(\mu, t)\rangle.\end{aligned} \tag{5.107}$$

In the limit $\mathrm{d}t \to 0$, (5.107) reduces to

$$\langle \dot{\boldsymbol{x}}(\nu, t)\rangle = (\hat{M}(\nu) - R)\langle \boldsymbol{x}(\nu, t)\rangle + R\sum_{\mu \neq \nu} p(\nu|\mu)\langle \boldsymbol{x}(\mu, t)\rangle. \tag{5.108}$$

If the number of values assumed by $\eta(t)$ are N then (5.108) constitutes a set of N equations whose elements are operators. The simplest telegraph process is the two-state one assuming values a, b with $p_{ab} = p_{ba} = 1$. As an example, see [71] where two-state telegraph noise is used as a model of fluctuations in the phase of a laser interacting with a two-level atom.

6. The Electromagnetic Field

In this chapter we recapitulate the classical and the quantum theories of the electromagnetic (e.m.) field. This is followed by a discussion of the concept of the field correlation functions and their role in characterizing the statistical and spectral properties of the field. We highlight some signatures of nonclassical features of the field carried by the correlation functions.

6.1 Free Classical Field

The e.m. field may be described by the vector potential $\boldsymbol{A}(\boldsymbol{r},t)$ and the scalar potential $\phi(\boldsymbol{r},t)$ related with the electric and the magnetic fields $\boldsymbol{E}(\boldsymbol{r},t)$ and $\boldsymbol{B}(\boldsymbol{r},t)$ by the relations

$$\boldsymbol{B}(\boldsymbol{r},t) = \nabla \times \boldsymbol{A}(\boldsymbol{r},t), \qquad \boldsymbol{E}(\boldsymbol{r},t) = -\frac{1}{\mathrm{c}}\frac{\partial}{\partial t}\boldsymbol{A}(\boldsymbol{r},t) - \nabla\phi(\boldsymbol{r},t). \tag{6.1}$$

The potential functions corresponding to particular electric and magnetic field vectors are not unique. They are arbitrary up to the gauge transformation $\boldsymbol{A} \to \boldsymbol{A} - \nabla\chi$, $\phi \to \phi + c^{-1}\partial\chi/\partial t$. We may use the freedom available within the gauge transformation to work with potentials having desired properties. The gauge convenient for developing the quantum theory of the e.m. field is the *Coulomb gauge* characterized by the *transversality condition* on $\boldsymbol{A}$:

$$\nabla.\boldsymbol{A}(\boldsymbol{r},t) = 0. \tag{6.2}$$

The potentials in a region of charge density σ and current density $\boldsymbol{j}$ under this condition obey the equations

$$\nabla^2\phi(\boldsymbol{r},t) = -\sigma, \tag{6.3a}$$

$$\nabla^2\boldsymbol{A}(\boldsymbol{r},t) - \frac{1}{\mathrm{c}^2}\frac{\partial^2}{\partial t^2}\boldsymbol{A}(\boldsymbol{r},t) = -\frac{1}{\mathrm{c}}\boldsymbol{j}_T. \tag{6.3b}$$

Here, $\boldsymbol{j} = \boldsymbol{j}_L + \boldsymbol{j}_T$ with the longitudinal and the transverse currents, $\boldsymbol{j}_L$ and $\boldsymbol{j}_T$ defined by $\nabla \times \boldsymbol{j}_L = 0$ and $\nabla.\boldsymbol{j}_T = 0$.

Consider a region free of charges and currents. In such a region, the inhomogenous terms in (6.3a) and (6.3b) vanish. We may exploit the Gauge freedom to choose $\phi = 0$ as the solution of (6.3a) corresponding to $\sigma = 0$. In order to solve (6.3b), we note that, for the boundary conditions of interest,

∇^2 is a hermitian operator. Hence, its eigenfunctions constitute a complete orthonormal set. Let $\{\boldsymbol{u}_k(\boldsymbol{r})\}$ be the eigenfunctions of ∇^2, called the *spatial mode functions*. They solve the eigenvalue equation

$$\nabla^2 \boldsymbol{u}_k(\boldsymbol{r}) + |\boldsymbol{k}|^2 \boldsymbol{u}_k(\boldsymbol{r}) = 0, \tag{6.4}$$

along with appropriate boundary conditions. Their orthonormality relation reads

$$\int_V \boldsymbol{u}_k^*(\boldsymbol{r}).\boldsymbol{u}_l(\boldsymbol{r}) \mathrm{d}^3 r = \delta_{kl} \tag{6.5}$$

where the integration is over the volume V of the region containing the field. We may then express $\boldsymbol{A}(\boldsymbol{r}, t)$ as

$$\boldsymbol{A}(\boldsymbol{r}, t) = \mathrm{c} \sum_k \frac{1}{\sqrt{2\omega_k}} \left[\boldsymbol{u}_k(\boldsymbol{r}) \alpha_k(t) + \boldsymbol{u}_k^*(\boldsymbol{r}) \alpha_k^*(t)\right]. \tag{6.6}$$

The form of the expansion coefficients above is chosen for later convenience. Note that the mode functions, by virtue of the transversality condition (6.2) on $\boldsymbol{A}(\boldsymbol{r}, t)$, should obey also the transversality condition

$$\nabla.\boldsymbol{u}_k(\boldsymbol{r}) = 0. \tag{6.7}$$

For example, in a cubic region of side $L = (V)^{1/3}$ with periodic boundary conditions, the mode functions are the *plane-waves*

$$\boldsymbol{u}_k(\boldsymbol{r}) \equiv \boldsymbol{u}_{\boldsymbol{k},\lambda}(\boldsymbol{r}) = \sqrt{\frac{1}{V}} \boldsymbol{\epsilon}_\lambda(\boldsymbol{k}) \exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{r}), \tag{6.8}$$

where $\boldsymbol{\epsilon}_\lambda(\boldsymbol{k})$ is called the *polarization vector*. As a result of (6.7), $\boldsymbol{\epsilon}_\lambda(\boldsymbol{k}) \cdot \boldsymbol{k} = 0$. Hence, for a given $\boldsymbol{k}$, $\boldsymbol{\epsilon}_\lambda(\boldsymbol{k})$ may be along any direction orthogonal to $\boldsymbol{k}$. Let $\boldsymbol{\epsilon}_1(\boldsymbol{k})$ and $\boldsymbol{\epsilon}_2(\boldsymbol{k})$ be the directions orthogonal to $\boldsymbol{k}$ and to each other such that $(\boldsymbol{\epsilon}_1(\boldsymbol{k}), \boldsymbol{\epsilon}_2(\boldsymbol{k}), \boldsymbol{k}/|\boldsymbol{k}|)$ forms a right-handed triad of mutually orthogonal directions. The eigenvector index k on $\boldsymbol{u}_k$ in this case stands for the set $\boldsymbol{k}, \lambda$.

We determine the expansion coefficients $\alpha_k(t)$ in (6.6) by substituting it in (6.3b) (with $\boldsymbol{j} = 0$). We find that the $\alpha_k(t)$ obey the equation of evolution of a harmonic oscillator:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \alpha_k(t) + \omega_k^2 \alpha_k(t) = 0. \qquad \omega_k = c|\boldsymbol{k}|. \tag{6.9}$$

We work with its solution

$$\alpha_k(t) = \exp(-\mathrm{i}\omega_k t) \alpha_k \tag{6.10}$$

so that, with $\boldsymbol{u}_k$ given by (6.8), $\boldsymbol{A}$ is a function of $\boldsymbol{k} \cdot \boldsymbol{r} - \omega_k t$ describing a travelling plane wave:

$$\boldsymbol{A}(\boldsymbol{r}, t) = \sum_k \frac{c}{\sqrt{2\omega_k}} \left[\boldsymbol{u}_k(\boldsymbol{r}) \alpha_k \exp(-\mathrm{i}\omega_k t) + \boldsymbol{u}_k^*(\boldsymbol{r}) \alpha_k^* \exp(\mathrm{i}\omega_k t)\right]. \tag{6.11}$$

The expression for the electric field, obtained by combining this with (6.1), reads

$$\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E}^{(+)}(\boldsymbol{r},t) + \boldsymbol{E}^{(-)}(\boldsymbol{r},t), \tag{6.12}$$

$$\boldsymbol{E}^{(+)}(\boldsymbol{r},t) = \sum_k \boldsymbol{E}_k^{(+)}(\boldsymbol{r},t),$$

$$\boldsymbol{E}_k^{(+)}(\boldsymbol{r},t) = \mathrm{i}\sqrt{\frac{\omega_k}{2}}\boldsymbol{u}_k(\boldsymbol{r})\alpha_k \exp(-\mathrm{i}\omega_k t) = \boldsymbol{E}_k^{(-)*}(\boldsymbol{r},t) \tag{6.13}$$

The $\boldsymbol{E}^{(\pm)}(\boldsymbol{r},t)$ are the so called *positive* and *negative* frequency parts of the field. We will see that separation of the electric field into positive and negative frequency parts plays a central role in the theory of the e.m. field detection.

Now, we recall that the hamiltonian of the e.m. field is given by

$$H = \frac{1}{2}\int_V \left[|\boldsymbol{E}|^2 + |\boldsymbol{B}|^2\right] \mathrm{d}^3 r. \tag{6.14}$$

This may be evaluated using the expression (6.13) for $\hat{E}$ and that for $\hat{B}$, obtained by combining (6.11) and (6.1). Using the orthogonality properties of $\boldsymbol{u_k}$ and the boundary conditions, it may be shown that

$$H = \sum_k \omega_k |\alpha_k|^2. \tag{6.15}$$

Refer to [73] for details. Introduce real variables q_k, p_k defined by

$$\alpha_k = \frac{\omega_k q_k + \mathrm{i}p_k}{\sqrt{2\omega_k}}. \tag{6.16}$$

Consequently, the hamiltonian (6.15) assumes the form

$$H = \frac{1}{2}\sum_k \left[p_k^2 + \omega^2 q_k^2\right]. \tag{6.17}$$

By treating them as canonically conjugate generalized dynamical variables, we note that the classical Hamilton's equation of q_k, p_k are, in accordance with their definition, the equations of a harmonic oscillator. The form of its hamiltonian (6.17) suggests that the free field may be regarded as a collection of harmonic oscillators. It affords an easy passage to the quantum theory of the radiation.

6.2 Field Quantization

The quantum theory of the e.m. field can be formulated by following the standard method of quantizing a classical system, namely, by treating the canonically conjugate variables q_k, p_k for each mode as operators $\hat{q}_k, \hat{p}_k$ which obey the commutation relation

$$\left[\hat{q}_k, \hat{p}_l\right] = \mathrm{i}\hbar\delta_{kl}. \tag{6.18}$$

The quantum analog of the classical free-field hamiltonian then is

$$\hat{H} = \frac{1}{2}\sum_k \left[\hat{p}_k^2 + \omega_k^2 \hat{q}_k^2\right]. \tag{6.19}$$

Define the creation and the annihilation operators

$$\hat{a}_k = \frac{\omega_k \hat{q}_k + \mathrm{i}\hat{p}_k}{\sqrt{2\hbar\omega_k}}, \qquad \hat{a}_k^\dagger = \frac{\omega_k \hat{q}_k - \mathrm{i}\hat{p}_k}{\sqrt{2\hbar\omega_k}}, \tag{6.20}$$

obeying, by virtue of (6.18), the commutation relation

$$\left[\hat{a}_k, \hat{a}_l^\dagger\right] = \delta_{kl}. \tag{6.21}$$

We may rewrite the Hamiltonian (6.19), after shifting the zero of the energy of the k^{th} oscillator to $\hbar\omega_k/2$, as

$$\hat{H} = \hbar \sum_k \omega_k \hat{a}_k^\dagger \hat{a}_k. \tag{6.22}$$

On comparing this with (6.15), we note quantum classical correspondence

$$\alpha_k \to \sqrt{\hbar}\hat{a}_k, \qquad \alpha_k^* \to \sqrt{\hbar}\hat{a}_k^\dagger. \tag{6.23}$$

The electric field is similarly quantized by replacing the c-number dynamical variables in (6.12) by operators. On invoking also (6.20), the electric field operator reads

$$\hat{\boldsymbol{E}}(\boldsymbol{r}, t) = \hat{\boldsymbol{E}}^{(+)}(\boldsymbol{r}, t) + \hat{\boldsymbol{E}}^{(-)}(\boldsymbol{r}, t), \tag{6.24}$$

$$\hat{\boldsymbol{E}}^{(+)}(\boldsymbol{r}, t) = \sum_k \hat{\boldsymbol{E}}_k^{(+)}(\boldsymbol{r}, t),$$

$$\hat{\boldsymbol{E}}_k^{(+)}(\boldsymbol{r}, t) = \mathrm{i}\sqrt{\frac{\hbar\omega_k}{2}}\boldsymbol{u}_k(\boldsymbol{r})\hat{a}_k \exp(-\mathrm{i}\omega_k t) = \hat{\boldsymbol{E}}_k^{(-)\dagger}(\boldsymbol{r}, t) \tag{6.25}$$

Now, recall that the eigenstates of $\hat{a}_k^\dagger \hat{a}_k$ are the number states $|n_k\rangle$ ($n_k = 0, 1, \cdots$. Hence energy of the field in the state $|n_k\rangle$ is $n_k\hbar\omega_k$. Since the energy in a field mode is an integral multiple of $\hbar\omega_k$, it is postulated that each unit of energy in a field mode is carried by an indestructible particle, called *photon*. Accordingly, an eigenstate $|\{n_k\}\rangle \equiv |n_1, n_2 \cdots\rangle$ of (6.22) represents a collection of photons with n_k photons in mode k. The ground state $|\{0_k\}\rangle$, which describes a field having no photon in any of the modes accessible to it, is called the *vacuum state* of the field. Invoke the properties of the h.o. operators given in Chap. 3 to verify that

$$\left\langle n_k|\hat{\boldsymbol{E}}_k|n_k\right\rangle = 0,$$

$$\left\langle n_k|\hat{\boldsymbol{E}}_k \cdot \hat{\boldsymbol{E}}_k|n_k\right\rangle - \left\langle n_k|\hat{\boldsymbol{E}}_k|n_k\right\rangle . \left\langle n_k|\hat{\boldsymbol{E}}_k|n_k\right\rangle = \frac{\hbar\omega_k}{V}\left(n_k + \frac{1}{2}\right). \tag{6.26}$$

This shows that the expectation value of electric field in the number state is zero but its fluctuations are finite even when it is in the vacuum state. These are the so called *vacuum fluctuations.*

Some particle-like properties of a photon are (see [73] for details):

- By invoking the expression

$$\boldsymbol{p} = \frac{1}{c}\int (\boldsymbol{E} \times \boldsymbol{B}) \mathrm{d}^3 r, \tag{6.27}$$

it may be shown that the momentum of a photon corresponding to the plane wave (6.8) of wave vector $\boldsymbol{k}$ is $\hbar\boldsymbol{k}$.

- On using the relativistic relation between mass m and energy E, namely, $\mathrm{mc}^2 = \sqrt{E^2 - |\boldsymbol{p}|^2 c^2}$ along with already noted facts that the momentum and energy of a photon of wavevector $\boldsymbol{k}$ are $\boldsymbol{p} = \hbar\boldsymbol{k}$ and $E = \hbar\omega_k = \mathrm{c}\hbar|\boldsymbol{k}|$, it follows that the mass of a photon is zero.
- Besides energy and momentum, a photon state is characterized also by the polarization vector $\boldsymbol{\epsilon}_\lambda$. Since $\boldsymbol{\epsilon}_\lambda$ transforms like a vector, a photon is viewed as a particle whose rotational properties are those of an angular momentum quantum number $L = 1$. Any component of such an angular momentum has eigenvalues $\pm 1, 0$. However, due to the transversality condition, the eigenvalue zero is excluded.

The quantum theory thus assigns particle-like character to what is classically perceived as a wave. Recall that the quantum theory predicts wave-like behaviour for what are perceived classically as particles. The quantum characteristics of the e.m. field are identified by comparing its statistical properties predicted by the quantum and classical theories outlined in the next two sections.

6.3 Statistical Properties of Classical Field

The classical electric field is expressible in terms of the spatial mode functions as in (6.13). The functions $\alpha_k(t)$ for free fields are harmonic. The same expansion holds also in the presence of sources except that the time-dependence of $\alpha_k(t)$ is determined by that of the source through (6.3b). The e.m. field may thus be generally characterized by a set of complex numbers $\{\alpha_k(t)\}$. For the sake of simplicity, we assume that the field is polarized in one direction and treat it as a scalar.

Detailed considerations based on the theory of atom-field interaction show that the optical detectors are square law detectors, i.e. they respond to the square of the field amplitude $E(\boldsymbol{r}, t)$ and that their time of resolution is orders of magnitude longer than the time period ω^{-1} of optical oscillations [74]. An optical detector, therefore, measures $E^2(\boldsymbol{r}, t)$ averaged over many cycles of oscillations. The detectors can be so arranged as to measure an nth order correlation function [74]

$$G^{(n)}\Big(\{\boldsymbol{r}_i t_i\}_n; \{\boldsymbol{r}_{i+n} t_{i+n}\}_n\Big) = \prod_{i=1}^{n} E^{(-)}(\boldsymbol{r}_i t_i)\, E^{(+)}(\boldsymbol{r}_{n+i} t_{n+i})$$
$$\equiv G^{(n)}\big(\boldsymbol{r}_1 t_1, \ldots, \boldsymbol{r}_n t_n; \boldsymbol{r}_{n+1} t_{n+1}, \ldots, \boldsymbol{r}_{2n} t_{2n}\big). \tag{6.28}$$

The first order correlation function $G^{(1)}(\boldsymbol{r}t,\boldsymbol{r}t)$ is a measure of intensity:

$$I(\boldsymbol{r}t) = |E^{(+)}(\boldsymbol{r}t)|^2 = G^{(1)}(\boldsymbol{r}t,\boldsymbol{r}t)\,. \tag{6.29}$$

In many situations of practical interest, field is a random function of time. Recall that the theory of random functions predicts only the averages over its many realizations. Now, the response time of optical detectors is usually much longer compared with the time over which the field changes. Consequently, the output of a detector at a given time is a result of an average of the function of the field being measured over the response time of the detector. In such a situation we can invoke the *ergodic theorem*, according to which, the average of a function of a random variable over a long period is the same as its average over all possible realizations of the values of the random variable. As explained in Chap. 5, the latter averages are characterized by the multi-time distribution functions. Accordingly, the correlation function (6.28) is determined by the average

$$G^{(n)}(\{\boldsymbol{r}_i t_i\}_n;\{\boldsymbol{r}_i t_i\}_n) = \left\langle \prod_{i=1}^{n} E^{(-)}(\boldsymbol{r}_i t_i)\, E^{(+)}(\boldsymbol{r}_{n+i} t_{n+i}) \right\rangle. \tag{6.30}$$

over an appropriate probability density characterizing the field. Note that, if $\boldsymbol{r}_{n+i} = \boldsymbol{r}_i$, $t_{n+i} = t_i$ then

$$G^{(n)}(\{\boldsymbol{r}_i t_i\}_n;\{\boldsymbol{r}_i t_i\}_n) = \left\langle \prod_{i=1}^{n} I(\boldsymbol{r}_i t_i) \right\rangle \geq 0. \tag{6.31}$$

This function determines the correlation between intensities at different space-time points.

Non-negativity of the probabilities leads to a number of inequalities between the correlation functions. As we will see below, inequalities play important role in identifying signatures of purely quantum effects. To that end, note that since probability densities are positive, $\langle|\Phi(\{x\})|^2\rangle \geq 0$ for any $\Phi(\{x\})$. Let $\Phi(x) = F_1(\{x\}) + \lambda F_2(\{x\})$ and show, following the method outlined in Sect. 1.1 for the derivation the Schwarz inequality that

$$|\langle F_1^* F_2\rangle|^2 \leq \langle|F_1|^2\rangle\langle|F_2|^2\rangle|. \tag{6.32}$$

Now, let

$$F_1 = E^{(+)}(\boldsymbol{r}_1 t_1)\cdots E^{(+)}(\boldsymbol{r}_n t_n)\,,$$
$$F_2 = E^{(+)}(\boldsymbol{r}_{n+1} t_{n+1})\cdots E^{(+)}(\boldsymbol{r}_{2n} t_{2n})$$

and apply (6.32) to show that

$$\left|G^{(n)}(\{\boldsymbol{r}_i t_i\}_n;\{\boldsymbol{r}_{i+n} t_{i+n}\}_n)\right|^2$$
$$\leq G^{(n)}(\{\boldsymbol{r}_i t_i\}_n;\{\boldsymbol{r}_i t_i\}_n)\, G^{(n)}(\{\boldsymbol{r}_{n+i} t_{i+n}\}_n;\{\boldsymbol{r}_{i+n} t_{i+n}\}_n)\,. \tag{6.33}$$

It turns out to be useful to introduce *normalized correlation functions*

$$g^{(n)}(\{\boldsymbol{r}_i t_i\}_n;\{\boldsymbol{r}_{i+n} t_{i+n}\}_n) = \frac{G^{(n)}(\{\boldsymbol{r}_i t_i\}_n;\{\boldsymbol{r}_{i+n} t_{i+n}\}_n)}{\prod_{i=1}^{2n}\sqrt{G^{(1)}(\boldsymbol{r}_i t_i;\boldsymbol{r}_i t_i)}}. \tag{6.34}$$

The correlation functions of frequent occurrence in quantum optics are the first and the second-order ones. In the following subsections we summarize briefly some of their important properties.

6.3.1 First-Order Correlation Function

The first order correlation function characterizes the interference effects in wave amplitudes like in the Young's double-slit and many other interferometric experiments. In order to see it, recall that in Young's double-slit experiment, the wave amplitude is divided in to two parts at two pinholes at positions $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ (see Fig. 6.1). The intensity of the field is measured

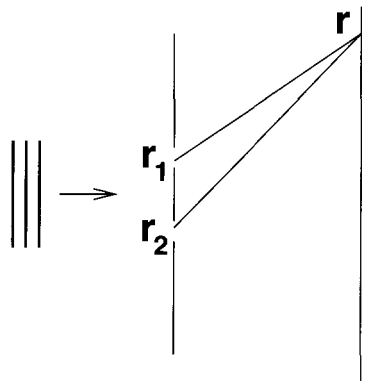

Fig. 6.1. Schematic diagram of Young's two-slit interference experiment.

at points $\boldsymbol{r}$ on another screen at a time t. The field $E(\boldsymbol{r}t)$ at $(\boldsymbol{r}, t)$ is due to superposition of the fields $\boldsymbol{E}(\boldsymbol{r}_i, t_i)$ $(i = 1, 2)$ that propagated from the two pinholes at earlier times t_i. Hence, the field intensity $I(\boldsymbol{r}, t)$ at the point of observation is

$$\begin{aligned} I(\boldsymbol{r}, t) &= |f_1 E^{(+)}(\boldsymbol{r}_1 t_1) + f_2 E^{(+)}(\boldsymbol{r}_2 t_2)|^2 \\ &= |f_1|^2 I(\boldsymbol{r}_1 t_1) + |f_2|^2 I(\boldsymbol{r}_2 t_2) \\ &\quad + 2\mathrm{Re}\left[f_1^* f_2 G^{(1)}(\boldsymbol{r}_1 t_1; \boldsymbol{r}_2 t_2)\right], \end{aligned} \tag{6.35}$$

the f_i being the constants inversely proportional to $|\boldsymbol{r} - \boldsymbol{r}_i|$. An important characteristic of the interference pattern is its *visibility* defined by

$$V = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} = \frac{2|f_1||f_2||G^{(1)}(\boldsymbol{r}_1 t_1; \boldsymbol{r}_2 t_2))|}{|f_1|^2 I(\boldsymbol{r}_1 t_1) + |f_2|^2 I(\boldsymbol{r}_2 t_2)}. \tag{6.36}$$

On invoking (6.33) and on combining it with the definition (6.34) of $g^{(1)}$, it follows that the visibility is maximum when

$$|g^{(1)}(\boldsymbol{r}_1 t_1; \boldsymbol{r}_2 t_2)| = 1. \tag{6.37}$$

The fields for which (6.37) holds are said to be *first order coherent.* The field is *incoherent* if $g^{(1)}(\boldsymbol{r}_1 t_1; \boldsymbol{r}_2 t_2) = 0$. The visibility with incoherent fields is zero, i.e. such fields do not produce any interference pattern. The field is said to be *partially coherent* if $|g^{(1)}(\boldsymbol{r}_1 t_1; \boldsymbol{r}_2 t_2))| \neq 0, 1$. It may be proved that (6.37) would hold for all space-time points if [74]

$$G^{(1)}(\boldsymbol{r}_1 t_1; \boldsymbol{r}_2 t_2) = \varepsilon^*(\boldsymbol{r}_1 t_1)\, \varepsilon(\boldsymbol{r}_2 t_2) \tag{6.38}$$

If, in addition, the field is stationary then $G^{(1)}(\boldsymbol{r}_1 t_1; \boldsymbol{r}_1 t_2)$ will be a function only of $t_1 - t_2 = \tau$. The condition (6.38) for such a process implies that $\varepsilon^*(\boldsymbol{r}_1 t_1)\varepsilon(\boldsymbol{r}_2 t_2)$ be a function of $t_1 - t_2$. This can hold only if $\varepsilon(t) \sim \exp(-i\omega t)$ implying thereby that a stationary first-order coherent field is monochromatic.

6.3.2 Second-Order Correlation Function

We have seen that the first-order correlation function characterizes interference effects between the field amplitudes at two space-time points. The equation (6.31) shows that $G^{(2)}(\boldsymbol{r}_1, t_1, \boldsymbol{r}_2, t_2; \boldsymbol{r}_2, t_2, \boldsymbol{r}_1, t_1)$, a second-order correlation function, is a measure of intensity correlations at two space-time points. The forerunner of the experiments for measuring intensity correlations is the one performed by Hanbury Brown and Twiss [75].

Consider the measurement of $G^{(2)}(\boldsymbol{r}t, \boldsymbol{r}t+\tau; \boldsymbol{r}t+\tau, \boldsymbol{r}t)$ assuming the field to be stationary. The correlation function in question is then a function only of τ. We denote it by $G^{(2)}(\tau)$. Its normalized form is

$$\begin{aligned} g^{(2)}(\tau) &= \frac{\langle E^{(-)}(t)E^{(-)}(t+\tau)E^{(+)}(t+\tau)E^{(+)}(t)\rangle}{\langle E^{(-)}(t)E^{(+)}(t)\rangle^2} \\ &= \frac{\langle I(t)I(t+\tau)\rangle}{\langle I(t)\rangle^2} \geq 0. \end{aligned} \tag{6.39}$$

The inequality above is a result of the fact that the intensity and the probability density are non-negative. Furthermore, the inequality (6.32) with $F_1 = I(t)$ and $F_2 = I(t+\tau)$ implies that

$$g^{(2)}(\tau) \leq g^{(2)}(0). \tag{6.40}$$

Next, let $F_1 = I(t)$ and $F_2 = 1$ in (6.32) to obtain

$$g^{(2)}(0) \geq 1. \tag{6.41}$$

This will turn out to be a useful inequality for distinguishing the quantum nature of the field.

6.3.3 Higher-Order Correlations

The higher-order correlation functions can be similarly interpreted. Of particular interest is the generalization of the definition of first-order coherence. A field is said to be n^{th} order coherent if [74]

$$\left|g^{(m)}\left(\{\boldsymbol{r}_i t_i\}_n;\{\boldsymbol{r}_{i+n} t_{i+n}\}_n\right)\right| = 1 \tag{6.42}$$

for all $m \leq n$. It may be shown that this would hold if [74]

$$G^{(m)}\left(\{\boldsymbol{r}_i t_i\}_n;\{\boldsymbol{r}_{i+n} t_{i+n}\}_n\right) = \prod_{i=1}^{m} \varepsilon^*(\boldsymbol{r}_i t_i) \prod_{i=m+1}^{2m} \varepsilon(\boldsymbol{r}_i t_i). \tag{6.43}$$

As a consequence of (6.31), we note that if the field is n^{th} order coherent then

$$g^{(m)}\left(\{\boldsymbol{r}_i t_i\}_n;\{\boldsymbol{r}_i t_i\}_n\right) = 1, \quad (m \leq n). \tag{6.44}$$

A field which is coherent to all orders is said to be *coherent.*

Next, we examine two commonly encountered examples of fields for their coherence properties.

6.3.4 Stable and Chaotic Fields

The simplest example is of a single-mode field with constant amplitude E_0 and phase ϕ:

$$E^{(+)}(t) = E_0 \exp[\mathrm{i}(\boldsymbol{k}.\boldsymbol{r} - \omega t + \phi)]. \tag{6.45}$$

Such a *stable classical field* describes approximately the output from a laser operating far above its threshold. For this field, the condition (6.42) of coherence holds for all n.

Consider now the field obtained by superposition of radiation from N radiators. For the sake of definiteness, let the radiators be atoms each emitting spontaneously at frequency ω_0. The atoms may be colliding with each other as well. The radiation from each atom is of constant amplitude e_0 but its phase is a random function of time. The positive frequency part of the total field at a point of observation is

$$E^{(+)}(t) = e_0 \sum_{i=1}^{N} \exp(-\mathrm{i}\omega_0 t + \mathrm{i}\phi_i(t)) \equiv \sum_{i=1}^{N} E_i^{(+)}(t) \tag{6.46}$$

The phase $\phi_i(t)$ is distributed uniformly between 0 and 2π. The corresponding distribution function being

$$f(\phi_1, \ldots, \phi_N) = \left(\frac{1}{2\pi}\right)^N. \tag{6.47}$$

We further assume that each $\phi_i(t)$ undergoes a sudden change at randomly distributed times but remains constant between two consecutive interruptions and that there is no correlation between the phase of radiation from different atoms. These characteristics are adequately described by the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_j(t) = \mu_j(t), \tag{6.48}$$

where $\mu_j(t)$ is a Gaussian white noise with zero mean and

$$\langle \mu_i(t+\tau)\mu_j(t)\rangle = 2\gamma\delta_{ij}. \tag{6.49}$$

The rate γ is related with the rate of spontaneous decay and the rate of collisions. Note that (6.48) and stationarity imply

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\exp\left[-\mathrm{i}\left\{\phi_i-\phi_i(\pm\tau)\right\}\right] = \pm\mathrm{i}\mu_i(\tau)\exp\left[-\mathrm{i}\left\{\phi_i-\phi_i(\pm\tau)\right\}\right]. \tag{6.50}$$

On recalling the deliberations of Chap. 5, we find that, with $\tau \geq 0$,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\langle\exp\left[-\mathrm{i}\left\{\phi_i-\phi_i(\pm\tau)\right\}\right]\right\rangle = -\gamma\left\langle\exp\left[-\mathrm{i}\left\{\phi_i-\phi_i(\pm\tau)\right\}\right]\right\rangle. \tag{6.51}$$

This proves useful in evaluating two-time averages. For,

$$\begin{aligned}
&\langle E^{(-)}(t)E^{(+)}(t+\tau)\rangle \\
&= e_0^2\exp(-\mathrm{i}\omega_0\tau)\sum_{i,j=1}^{N}\left\langle\exp\left[-\mathrm{i}\left\{\phi_i(t)-\phi_j(t+\tau)\right\}\right]\right\rangle \\
&= e_0^2\exp(-\mathrm{i}\omega_0\tau)\sum_{i=1}^{N}\left\langle\exp\left[-\mathrm{i}\left\{\phi_i(t)-\phi_i(t+\tau)\right\}\right]\right\rangle \\
&= Ne_0^2\exp(-\mathrm{i}\omega_0\tau)\left\langle\exp\left[-\mathrm{i}\left\{\phi_i(t)-\phi_i(t+\tau)\right\}\right]\right\rangle \\
&= Ne_0^2\exp(-\mathrm{i}\omega_0\tau-\gamma|\tau|).
\end{aligned} \tag{6.52}$$

In writing the second line above we have invoked the fact that the atoms are not correlated enabling us to write the average of a product involving different atoms as a product of the averages. Also, by virtue of (6.47), the average of $\exp(\mathrm{i}\phi_j)$ is zero. The property of equivalence of the radiators has been applied in writing the third line. The final result is obtained by inserting the solution of (6.51). Verify also that, due to (6.47),

$$\langle E^{(-)}(t)E^{(-)}(t+\tau)\rangle = \langle E^{(+)}(t)E^{(+)}(t+\tau)\rangle = 0. \tag{6.53}$$

We recall again the considerations of Chap. 5 now to construct single time probability density $P(E,E^*)$ by evaluating

$$\begin{aligned}
P(E,E^*,t) = \left(\frac{1}{\pi}\right)^2\int \mathrm{d}^2\xi\exp\left[-\mathrm{i}\left(E\xi+E^*\xi^*\right)\right] \\
\times\left\langle\exp\left[\mathrm{i}\left(E^{(+)}(t)\xi+E^{(-)}(t)\xi^*\right)\right]\right\rangle.
\end{aligned} \tag{6.54}$$

With $E^{(+)}(t)$ given by (6.46), we find that

$$\begin{aligned}
&\left\langle\exp\left[\mathrm{i}(E^{(+)}(t)\xi+E^{(-)}(t)\xi^*)\right]\right\rangle \\
&= \left(\frac{1}{2\pi}\right)^N\prod_{j=1}^{N}\int_0^{2\pi}\mathrm{d}\phi_1\cdots\mathrm{d}\phi_N\exp\left[\mathrm{i}e_0\left\{\exp(\mathrm{i}\phi_j-\mathrm{i}\omega_0 t)\xi+\text{c.c.}\right\}\right]
\end{aligned}$$

$$= \left[\sum_{m=0}^{\infty} \frac{(-)^m}{(m!)^2} e_0^{2m} |\xi|^{2m}\right]^N$$
$$\approx \exp\left[-Ne_0^2|\xi|^2\right]. \tag{6.55}$$

The last line above is the result of retaining only the first two terms in the series in the second line and by invoking the definition of an exponential assuming $N \gg 1$. It is called the *Gaussian approximation.* On combining (6.54) and (6.55) and on applying (A.28), follows the *chaotic field distribution*

$$P(E, E^*) = \frac{1}{\pi e_0^2 N} \exp\left[-|E|^2/Ne_0^2\right]. \tag{6.56}$$

The average of an arbitrary product of field for this distribution is

$$\left\langle E^{(-)m} E^{(+)n} \right\rangle = \int d^2 E E^n E^{*m} P(E, E^*) = m! \left(Ne_0^2\right)^m \delta_{mn}. \tag{6.57}$$

The multi-time probabilities may similarly be approximated as Gaussians. Consequently, all the cumulants except the first two vanish. We exploit this property to evaluate $\langle E^{(-)}(t)E^{(-)}(t+\tau)E^{(+)}(t+\tau)E^{(+)}(t)\rangle$ by equating the cumulant $\langle\langle\langle E^{(-)}(t)E^{(-)}(t+\tau)E^{(+)}(t+\tau)E^{(+)}(t)\rangle\rangle\rangle$ to zero. We use the expression for the cumulant of four variables in terms of the moments (see, for example [65]). Since all odd-order moments are zero, we find that

$$\begin{aligned}
&\langle E^{(-)}(t)E^{(-)}(t+\tau)E^{(+)}(t+\tau)E^{(+)}(t)\rangle \\
&= \langle E^{(-)}(t)E^{(+)}(t)\rangle\langle E^{(-)}(t+\tau)E^{(+)}(t+\tau)\rangle \\
&\quad + \langle E^{(-)}(t)E^{(+)}(t+\tau)\rangle\langle E^{(-)}(t+\tau)E^{(+)}(t)\rangle.
\end{aligned} \tag{6.58}$$

This relation holds for any Gaussian with zero mean. On recalling (6.52), (6.58) yields

$$\langle E^{(-)}(t)E^{(-)}(t+\tau)E^{(+)}(t+\tau)E^{(+)}(t)\rangle = N^2 e_0^4 \left[1 + \exp(-2\gamma|\tau|)\right]. \tag{6.59}$$

Now, the normalized first-order correlation function for a stationary field at a fixed point of observation with $t_1 = t, t_2 = t + \tau$ is given, using (6.52) and (6.59), by

$$\begin{aligned}
g^{(1)}(\tau) &= \frac{\langle E^{(-)}(t)E^{(+)}(t+\tau)\rangle}{\langle E^{(-)}(t)E^{(+)}(t)\rangle} \\
&= \exp(-i\omega_0\tau - \gamma|\tau|).
\end{aligned} \tag{6.60}$$

The normalized second-order correlation function defined in (6.39), is given by

$$g^{(2)}(\tau) = 1 + \exp(-2\gamma|\tau|). \tag{6.61}$$

Consider first the properties of $g^{(1)}(\tau)$.

- For $\gamma \neq 0$, $|g^{(1)}(\tau)| \neq 0, 1$. Hence the chaotic field is partially coherent.

- For $\tau \ll \gamma^{-1}$, $|g^{(1)}(\tau)| \approx 1$, which is its value for a coherent field whereas for $\tau \gg \gamma^{-1}$, $g^{(1)}(\tau) \to 0$ which is its value for an incoherent field. Hence γ^{-1} is identified as the *coherence time* of the field.
- $g^{(1)}(0) = 1$. Hence $g^{(1)}(0)$ can not distinguish between a coherent and a chaotic field.

Consider now $g^{(2)}(\tau)$ given by (6.61). Note that

$$g^{(2)}(0) = 2, \tag{6.62}$$

which is in contrast with the value $g^{(2)}(0) = 1$ for a coherent field. The chaotic and the coherent fields can thus be distinguished by a measurement of $g^{(2)}(0)$. However, note from the expression (6.61) for the chaotic field that

$$g^{(2)}(\tau) \to 1, \qquad \tau \gg \gamma^{-1}. \tag{6.63}$$

Hence, on a time scale much longer than the coherence time, the chaotic and the coherent fields are indistinguishable.

Let us now express the field as in (6.13). Since we are restricting our attention to a fixed point ($\boldsymbol{r} = 0$) in space, we find that

$$\alpha(\omega_k) = \frac{C}{\sqrt{\omega_k}} \int_{-\infty}^{\infty} \exp(\mathrm{i}\omega_k t) E^{(+)}(t) \mathrm{d}t, \tag{6.64}$$

the C being a constant. This shows that $\langle \alpha(\omega_k) \rangle = 0$. Following the steps in (6.89), and on redefining the constant C, we obtain

$$\langle \alpha(\omega_k) \alpha(\omega_l) \rangle = \frac{C}{\omega_k} \delta(\omega_k - \omega_l) S(\omega_k), \tag{6.65}$$

$$S(\omega) = \frac{2\gamma}{\gamma^2 + (\omega - \omega_0)^2}. \tag{6.66}$$

This shows that the frequency in a chaotic field has Lorentzian distribution. In view of (6.56), the set $\{\alpha_k, \alpha_k^*\}$ also has a Gaussian distribution. Their distribution function is, therefore, given by

$$P(\alpha_k, \alpha_k^*) = \prod_k \frac{1}{\pi \langle |\alpha(\omega_k)|^2 \rangle} \exp\left(-|\alpha_k|^2 / \langle |\alpha(\omega_k)|^2 \rangle\right). \tag{6.67}$$

We will use this result in the next section to construct the quantum analog of the classical chaotic field.

6.4 Statistical Properties of Quantized Field

The observable statistical properties of the e.m. field in the quantum theory are determined by the correlation function

$$G^{(n)}(\{\boldsymbol{r}_i t_i\}_n; \{\boldsymbol{r}_{i+n} t_{i+n}\})$$
$$= \left\langle \hat{E}^{(-)}(\boldsymbol{r}_1 t_1) \cdots \hat{E}^{(-)}(\boldsymbol{r}_n t_n) \right.$$
$$\left. \times \hat{E}^{(+)}(\boldsymbol{r}_{n+1} t_{n+1}) \cdots \hat{E}^{(+)}(\boldsymbol{r}_{2n} t_{2n}) \right\rangle. \tag{6.68}$$

Bear in mind that $\hat{E}^{(+)}(\boldsymbol{r}t) = \hat{E}^{(-)\dagger}(\boldsymbol{r}t)$ and that the average is the quantum mechanical expectation value with the density operator of the field. The expression (6.68) reduces to the corresponding classical one if operators are replaced by corresponding c-numbers and the averages are identified as over the distribution of those c-numbers. Since $\hat{E}^{(+)}$ is a linear combination of field annihilation operators for different modes and $\hat{E}^{(-)} = \hat{E}^{(+)\dagger}$ is that of field creation operators, (6.68) shows that the process of photo detection measures the field operators in normal ordering. Recall that it is this fact that forms the basis for classifying the field states in Chap. 4 as classical or quantum.

Now, purely quantum effects may be isolated by comparing the bounds on correlation functions derived in the last section for the classical theory with those predicted by the quantum theory, to be derived below. To that end, recall from Sect. 1.1 that $\hat{C}^\dagger \hat{C}$ for any $\hat{C}$ is a positive operator. Since density operator is also positive, it follows that $\mathrm{Tr}[\hat{C}^\dagger \hat{C} \hat{\rho}] \geq 0$. Let $\hat{C} = \hat{A} + \lambda \hat{B}$. Follow the method outlined in Sect. 1.1 for the derivation the Schwarz inequality and show that

$$|\langle \hat{A}^\dagger \hat{B} \rangle|^2 \leq \langle \hat{A}^\dagger \hat{A} \rangle \langle \hat{B}^\dagger \hat{B} \rangle. \tag{6.69}$$

The choice

$$\hat{A} = \prod_{i=1}^{n} \hat{E}^{(+)}(\boldsymbol{r}_i t_i), \quad \hat{B} = \prod_{i=1}^{n} \hat{E}^{(+)}(\boldsymbol{r}_{i+n} t_{i+n}), \tag{6.70}$$

leads to the same inequality (6.33) as derived for a classical field. However, we show below that there are situations in which the quantum predictions differ from the predictions based on classical inequalities.

6.4.1 First-Order Correlation

The normalized first-order correlation function for a quantized field at a fixed point of observation and for a stationary process is defined as in (6.60) with the c-numbers replaced by operators. For a single-mode field it assumes the form

$$g^{(1)}(\tau) = \frac{\langle \hat{a}^\dagger(t) \hat{a}(t+\tau) \rangle}{\langle \hat{a}^\dagger(t) \hat{a}(t) \rangle}. \tag{6.71}$$

Quantum theoretical $g^{(1)}(\tau)$ does not lead to any inequality which is in conflict with the classical theory. This amounts to saying that the first-order correlation function does not carry any signature of field quantization.

6.4.2 Second-Order Correlation

Now we examine the properties of the quantized version of the second-order correlation function defined in (6.39):

$$g^{(2)}(\tau) = \frac{\langle \hat{E}^{(-)}(t)\hat{E}^{(-)}(t+\tau)\hat{E}^{(+)}(t+\tau)\hat{E}^{(+)}(t)\rangle}{\langle \hat{E}^{(-)}(t)\hat{E}^{(+)}(t)\rangle^2}. \tag{6.72}$$

For a single-mode field it reduces to

$$g^{(2)}(\tau) = \frac{\langle \hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau)\hat{a}(t+\tau)\hat{a}(t)\rangle}{\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle^2}. \tag{6.73}$$

Let $\tau = 0$ and reexpress (6.72) as

$$g^{(2)}(0) = \frac{\left\langle \left(\hat{E}^{(-)}\hat{E}^{(+)}\right)^2 \right\rangle}{\left\langle \hat{E}^{(-)}\hat{E}^{(+)} \right\rangle^2} - \frac{\left\langle \hat{E}^{(-)}\left[\hat{E}^{(+)},\ \hat{E}^{(-)}\right]\hat{E}^{(+)} \right\rangle}{\langle \hat{E}^{(-)}\hat{E}^{(+)}\rangle^2}. \tag{6.74}$$

On applying (6.69) with $\hat{A} = \hat{E}^{(-)}\hat{E}^{(+)}$, $\hat{B} = 1$ we see that the first term on the right hand side above is greater than one. However, non-zero positive values of the second term may lead to

$$g^{(2)}(0) < 1. \tag{6.75}$$

Notice that (6.75) can hold only if the commutator in the second term in (6.74) is non-zero. On recalling the classical prediction (6.74) viz. $g^{(2)}(0) \geq 1$, it follows that (6.75) is a signature of the quantum nature of the field. Furthermore, using the P-representation introduced in Chap. 3, we leave it to the reader to confirm that $g^{(2)}(0) < 1$ implies that the corresponding P-function is non-classical. A field for which (6.75) holds is called *antibunched*. An example of antibunched field is the number state $|m\rangle$. For this state, it is easy to see using (6.73) that $g^{(2)}(0) = 1$ for $m = 0$, and $g^{(2)}(0) = 1 - 1/m$ for $m \geq 1$.

Now, for a single-mode field, (6.73) and (6.75) imply

$$\langle (\hat{a}^\dagger\hat{a})^2\rangle - \langle \hat{a}^\dagger\hat{a}\rangle^2 \leq \hat{a}^\dagger\hat{a}. \tag{6.76}$$

The left hand side above is the variance and the right hand side the mean of photon number distribution. Recall that variance is equal to mean if the distribution is Poissonian. The inequality in (6.76) indicates *sub-Poissonian* photon number distribution. Hence, sub-Poissonian photon number distribution is another signature of the quantum nature of the e.m. field.

6.4.3 Quantized Coherent and Thermal Fields

In this subsection we identify the quantum analogs of classical stable and chaotic fields.

Like in the classical description, a quantum field is said to be n^{th} order coherent if (6.42) holds. It implies the factorization property (6.43). On using the operator representation of the field, it follows that the condition for coherence to all orders can be satisfied if the field is in an eigenstate of the annihilation operator $\hat{a}_k$ of every mode k (see [74] for details). Since the operators corresponding to different modes commute, it follows that the said eigenstate is a direct product of the eigenstates of each of the annihilation operator $\hat{a}_k$. An eigenstate $|\alpha\rangle$ of the field annihilation operator $\hat{a}$ is, therefore, called a *coherent state* of the field. This provides physics basis for the concept of a coherent state introduced in Chap. 3 based on mainly mathematical considerations. Recall also from Chap. 3 that $\hat{a}^\dagger$ does not admit right eigenstates. Hence the states $|\alpha\rangle$ are the only ones for which the correlation function factorizes to all orders.

The expansion (3.28) of a coherent state in terms of the number states and the interpretation of the number state $|m\rangle$ as the state having m photons shows that the photon number distribution in the coherent state is Poissonian. Verify using (6.24) that the expectation value of single-mode electric field in the coherent state $|\alpha\rangle$, $(\alpha = |\alpha| \exp(\mathrm{i}\phi))$ is given by

$$\langle \hat{E}_k^{(+)}(\boldsymbol{r}, t)\rangle = \mathrm{i}\sqrt{\frac{\hbar\omega_k}{2V}}\alpha \exp[\mathrm{i}(\boldsymbol{k}.\boldsymbol{r} - \omega_k t)]. \tag{6.77}$$

This expression is similar to the one for stable classical field. Also, like the stable classical field, the field in a coherent state is coherent to all orders. The field in a coherent state is therefore analogous to a stable classical field.

Next, we construct the quantum analog of the classical chaotic field. To that end, we invoke the correspondence (6.23) and note that the quantum density operator analogous to the classical distribution function (6.67) of the chaotic field may be written as

$$\hat{\rho} = \prod_k A_k \exp(-\beta_k \hat{a}_k^\dagger \hat{a}_k) \tag{6.78}$$

where β_k and the normalization constant A_k are determined as follows.

- Working in the basis of the eigenstates $|m\rangle$ of $\hat{a}^\dagger\hat{a}$ we obtain

$$\mathrm{Tr}\Big[\exp(-\beta\hat{a}^\dagger\hat{a})\Big] = \sum_{m=0}^{\infty} \exp(-m\beta) = \frac{1}{1 - \exp(-\beta)}. \tag{6.79}$$

Hence, the normalization constant A_k is given by

$$A_k = \prod_k [1 - \exp(-\beta_k)] \tag{6.80}$$

- The expectation value of $(\hat{a}^\dagger\hat{a})^m$ is similarly found to be given by

$$\mathrm{Tr}\Big[(\hat{a}^\dagger\hat{a})^m \exp(-\beta\hat{a}^\dagger\hat{a})\Big]$$

$$= (-)^m \frac{\mathrm{d}^m}{\mathrm{d}\beta^m} \mathrm{Tr}\Big[\exp(-\beta \hat{a}^\dagger \hat{a})\Big]$$
$$= (-)^m \frac{\mathrm{d}^m}{\mathrm{d}\beta^m} \frac{1}{1-\exp(-\beta)}. \tag{6.81}$$

Using this, it is straightforward to see that

$$\langle \hat{a}_k^\dagger \hat{a}_l \rangle = \bar{n}_k \delta_{kl}, \quad \bar{n}_k = \frac{\exp(-\beta_k)}{1-\exp(-\beta_k)}. \tag{6.82}$$

The correspondence (6.23) implies that $\hbar \langle \hat{a}_k^\dagger \hat{a}_l \rangle = \langle \alpha_k^* \alpha_l \rangle$. Use this relation, the expressions (6.65) and (6.82) to show that

$$\hbar \bar{n}_k = \frac{C}{\omega_k} S(\omega_k). \tag{6.83}$$

We may now evaluate normalized first order correlation function defined in (6.60) by inserting in it the expression (6.25) for the field. Bearing (6.82) and (6.83) in mind, verify that

$$g^{(1)}(\tau) = \frac{\sum_k \omega_k \bar{n}_k \exp(-\mathrm{i}\omega_k \tau)}{\sum_k \omega_k \bar{n}_k} = \frac{\sum_k S(\omega_k) \exp(-\mathrm{i}\omega_k \tau)}{\sum_k S(\omega_k)}. \tag{6.84}$$

Convert the sum to an integral over ω from 0 to ∞. As is the case with optical fields, the width γ of the Lorentzian is small compared with the optical frequencies. Hence, we can extend lower limit of integration to $-\infty$. On carrying the integration by standard contour integration, we recover the classical result (6.60). We leave it to the reader to evaluate $g^{(2)}(\tau)$ in the same way and confirm that it is same as (6.61) for the classical chaotic field. In particular, $g^{(2)}(0) = 2$. The value $g^{(2)}(0) = 2$ may now be interpreted to indicate increased correlation or *bunching* of photons at a given space-time point over what is found in a coherent state.

6.5 Homodyned Detection

In the last section we noted that a direct measurement on the field at a given space-time point yields information only about the quantities which are products of equal number of positive and negative frequency parts of the field operator. The observables which are not of the said form can be measured by the method of *homodyned detection.* In this method, the signal field is mixed with a strong coherent field called the *local oscillator.* Photo detection of the field so mixed can give information about the statistical properties of certain correlation functions of the kind in question. See [76] for details.

As an illustration, let the signal be a single mode field described by the operators $\hat{a}, \hat{a}^\dagger$. Let frequency of the local oscillator field be the same as that of the signal. We treat the local oscillator classically. The positive frequency part of the mixed fields is

$$\hat{E}^{(+)} = [\hat{a} + |E_l| \exp(-\mathrm{i}\phi)] \exp(-\mathrm{i}\omega t). \tag{6.85}$$

Here $|E_l|$ and ϕ are the amplitude and phase of the local oscillator. Perform a measurements of the variance in the photon number distribution in the homodyned field. Assume the local oscillator field to be very strong compared with the signal and retain the highest non-zero power of $|E_l|$ to show that

$$\Delta n^2 \approx 2|E_l|^2 \left[\langle \hat{A}_\phi^2 \rangle - \langle \hat{A}_\phi \rangle^2\right], \tag{6.86}$$

$$\hat{A}_\phi = \frac{1}{\sqrt{2}} \left[\exp(\mathrm{i}\phi)\hat{a} + \exp(-\mathrm{i}\phi)\hat{a}^\dagger\right]. \tag{6.87}$$

A measurement of Δn^2 in homodyned detection of a single-mode field is a measure of the variance in $\hat{A}_\phi$.

The method outlined above for a single-mode field can be extended to multi-mode fields by means of appropriate choice of frequency of the local oscillator. As an example, see [55].

6.6 Spectrum

A characteristic of the light fields of utmost interest has traditionally been its spectrum. By spectrum we commonly understand the intensity of the Fourier components of the field. Let

$$\tilde{E}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(\mathrm{i}\omega t) E^{(+)}(t) \mathrm{d}t. \tag{6.88}$$

Assume that the field is stationary. Now,

$$\begin{aligned}
&\langle \tilde{E}^*(\omega_1)\tilde{E}(\omega_2)\rangle \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}t_1 \int_{-\infty}^{\infty} \mathrm{d}t_2 \exp\{-\mathrm{i}(\omega_1 t_1 - \omega_2 t_2)\langle E^{(-)}(t_1)E^{(+)}(t_2)\rangle \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}T \exp\{-\mathrm{i}T(\omega_1 - \omega_2)\} \\
&\quad \times \int_{-\infty}^{\infty} \mathrm{d}\tau \exp\{\mathrm{i}\tau(\omega_1 + \omega_2)/2\}\langle E^{(-)}E^{(+)}(\tau)\rangle \\
&= \delta(\omega_1 - \omega_2) S(\omega_1).
\end{aligned} \tag{6.89}$$

The second line above is obtained by changing the integration variables to $T = (t_1 + t_2)/2$, $\tau = t_2 - t_1$ along with the use of the stationarity, $\langle E^{(-)}(t_1)E^{(+)}(t_2)\rangle = \langle E^{(-)}E^{(+)}(t_2 - t_1)\rangle$, and

$$\begin{aligned}
S(\omega) &= \int_{-\infty}^{\infty} \mathrm{d}\tau \exp(\mathrm{i}\omega\tau)\langle E^{(-)}E^{(+)}(\tau)\rangle \\
&= 2\mathrm{Re} \int_0^{\infty} \mathrm{d}\tau \exp(\mathrm{i}\omega\tau)\langle E^{(-)}E^{(+)}(\tau)\rangle.
\end{aligned} \tag{6.90}$$

The last line above is obtained by using the relation $\langle E^{(-)}E^{(+)}(-\tau)\rangle = \langle E^{(-)}(\tau)E^{(+)}\rangle = \langle E^{(-)}E^{(+)}(\tau)\rangle^*$. The function $S(\omega)$ is called the *spectral density* or the *spectrum* of the field. The equation (6.90), relating spectral density with the two-time correlation function of a stationary process, is the content of the *Wiener-Khintchine theorem.* For a plane-wave field, the mean power per unit area carried by the field past a point of observation is proportional to $\langle E^{(-)}E^{(+)}\rangle$, it follows that $S(\omega)$ is indeed a measure of the power distribution among different frequency components of the field[77]. The definition (6.90) may be extended to quantized fields by treating the field amplitude as an operator so that

$$S(\omega) = 2\mathrm{Re}\left[\int_0^\infty \mathrm{d}\tau \exp(\mathrm{i}\omega\tau)\left\langle \hat{E}^{(-)}\hat{E}^{(+)}(\tau)\right\rangle\right]. \tag{6.91}$$

Note that the definitions of spectrum given above are based on mathematical considerations. The mathematical soundness of these definitions, their relationship with what is actually measured as also the question of generalization to a non-stationary process have been greatly debated. We refer to [77, 78] for a detailed account of the theory of the spectrum of classical and quantized e.m. field. The definition (6.91), however, is adequate for most commonly encountered situations.

7. Atom–Field Interaction Hamiltonians

In this chapter we introduce the hamiltonian governing the interaction of quantized electromagnetic (e.m.) field with free atoms. We show how the facts of the physics of the problem can be used to reduce that hamiltonian to mathematically manageable forms. Simple prescriptions for constructing models conforming to frequently encountered conditions are given.

7.1 Dipole Interaction

Consider an atom consisting of Z electrons surrounding its nucleus which is assumed to be at rest. Let the atom be irradiated by the e.m. field. Express the hamiltonian of the combined system of the atom and the field as

$$\hat{H} = \hbar \sum_k \omega_k \hat{a}_k^\dagger \hat{a}_k + \hat{H}_{\rm a} + \hat{H}_{\rm int}. \tag{7.1}$$

The first term on the right hand side above is the free field hamiltonian, given by (6.22), $\hat{H}_{\rm a}$ and $\hat{H}_{\rm int}$ are, respectively, the free-atom and atom-field interaction hamiltonians derived below. If the wavelength of the field is large compared with the size of the atom, which is typically of the order of 10^{-8} c.m., then the variation of the e.m. field over the atom can be ignored. This would be the case if the field frequency is less than about 10^{18} Hz. It can then be shown that the dominant part of the atom-field interaction arises by treating the atom as a dipole constituted by Z atomic electrons bound to the nucleus [72]. If $\boldsymbol{r}_i$ $(i = 1, \cdots, Z)$ denotes the position of the $i^{\rm th}$ electron then the quantized atom-field interaction hamiltonian in the electric dipole approximation reads

$$\hat{H}_{\rm int} = -\hat{\boldsymbol{d}} \cdot \hat{\boldsymbol{E}}(\boldsymbol{r}_0) = -e \sum_{i=1}^{Z} \hat{\boldsymbol{r}}_i \cdot \hat{\boldsymbol{E}}(\boldsymbol{r}_0). \tag{7.2}$$

Here e is the electronic charge, $\hat{\boldsymbol{d}}$ is the atomic dipole moment, and $\hat{\boldsymbol{E}}(\boldsymbol{r}_0)$ is the electric field operator at the position $\boldsymbol{r}_0$ of the nucleus:

$$\hat{\boldsymbol{E}}(\boldsymbol{r}_0) = {\rm i} \sum_k \sqrt{\frac{\hbar\omega_k}{2V}} \boldsymbol{\epsilon}_\lambda(\boldsymbol{k}) \exp({\rm i}\boldsymbol{k} \cdot \boldsymbol{r}_0) \hat{a}_k + {\rm h.c.} \tag{7.3}$$

We assume that all but the valence electrons constitute a hard core and that the e.m. field changes the state of only the valence electrons. We consider atoms having single valence electron and restrict the summation in (7.2) to $Z = 1$.

Now, let $\{|\psi_i\rangle\}$, $i = 1, \cdots, N$ be the complete set of orthonormal energy eigenstates accessible to the electron and let $\{E_i\}$ be the corresponding energy eigenvalues. As a result, the relevant part of the *free-atom hamiltonian* reads

$$\hat{H}_{\rm a} = \sum_{i=1}^{N} E_i \hat{A}_{ii}, \tag{7.4}$$

$$\hat{A}_{ij} = |\psi_i\rangle\langle\psi_j|, \qquad (i, j = 1, \cdots, N). \tag{7.5}$$

The orthogonality and completeness of the states implies that

$$\hat{A}_{ij}\hat{A}_{kl} = \hat{A}_{il}\delta_{jk}, \qquad \sum_{i=1}^{N} \hat{A}_{ii} = 1. \tag{7.6}$$

The dipole moment operator in the basis of the energy eigenstates reads

$$\hat{\boldsymbol{d}} = \sum_{i,j=1}^{N} \boldsymbol{d}_{ij} |\psi_i\rangle\langle\psi_j|, \tag{7.7}$$

where $\boldsymbol{d}_{ij} = e\langle\psi_i|\hat{\boldsymbol{r}}|\psi_j\rangle$ is the dipole matrix element between the levels $|\psi_i\rangle$ and $|\psi_j\rangle$. In the representation spanned by the eigenstates $|\boldsymbol{r}\rangle$ of the position operator $\hat{\boldsymbol{r}}$, obeying $\hat{\boldsymbol{r}}|\boldsymbol{r}\rangle = \boldsymbol{r}|\boldsymbol{r}\rangle$,

$$\boldsymbol{d}_{ij} = e \int \mathrm{d}^3 r \boldsymbol{r} \psi_i^*(\boldsymbol{r}) \psi_j(\boldsymbol{r}), \quad \psi_k(\boldsymbol{r}) = \langle \boldsymbol{r}|\psi_k\rangle. \tag{7.8}$$

Verify that if the electronic states are of definite parity, i.e. if $\psi_k(-\boldsymbol{r}) = \pm\psi_k(\boldsymbol{r})$ then $\boldsymbol{d}_{ij} = 0$ if the parity of $|\psi_i\rangle$ is same as that of $|\psi_j\rangle$. For such states $\boldsymbol{d}_{ii} = 0$. Each state of a free atom has a definite parity. On combining (7.2), (7.3), and (7.7), the interaction hamiltonian assumes the form

$$\hat{H}_{\rm int} = \hbar \sum_{i,j=1}^{N} \sum_{k} \hat{A}_{ij} \left[g_{ijk} \hat{a}_k + g_{ijk}^* a_k^\dagger \right], \tag{7.9}$$

the g_{ijk} being the atom-field coupling constant given by

$$g_{ijk} = -\mathrm{i}\sqrt{\frac{\omega_k}{2\hbar V}}\, \boldsymbol{d}_{ij}.\boldsymbol{\epsilon}_\lambda \exp(\mathrm{i}\boldsymbol{k}.\boldsymbol{r}_0) \equiv \boldsymbol{d}_{ij}.\boldsymbol{\epsilon}_\lambda e_k. \tag{7.10}$$

If the matrix element of the dipole operator between two levels vanishes then the transition between those levels is said to be electric dipole forbidden. Else it is an electric dipole allowed transition. The selection rules for identifying electric dipole allowed transitions are well known.

The hamiltonian for an interacting system of an atom and the e.m. field in dipole approximation thus assumes the form

$$\hat{H} = \hbar \sum_{k=0} \omega_k \hat{a}_k^\dagger \hat{a}_k + \sum_{i=1}^{N} E_i \hat{A}_{ii} + \hbar \sum_{i,j=1}^{N} \sum_k \hat{A}_{ij} \left[g_{ijk} \hat{a}_k + g_{ijk}^* a_k^\dagger \right]. \quad (7.11)$$

It treats the atom as well as the field quantum mechanically. If we wish to identify the characteristic effects of field quantization then we need to compare the results of the quantized hamiltonian (7.11) with those predicted by treating the field classically. On recalling the quantum-classical correspondence (6.23), we note that the classical field version of the atom-field interaction is obtained by the replacement $\hat{a}_k \to \alpha_k/\sqrt{\hbar}$. The equations of the field variables are then the Hamilton's equations for the pairs of conjugate variables $\{\alpha_k, \alpha_k^*\}$. Those equations are the same as the Maxwell's equations. The approximation which treats the e.m. field classically but the atoms quantum mechanically is known as the *semiclassical approximation*. It is equivalent with replacing the quantum averages $\langle \hat{a}^{\dagger m} \hat{a}^n \hat{A} \rangle$, where $\hat{A}$ is an atomic operator, by $\langle \hat{a}^{\dagger m} \rangle \langle a^n \rangle \langle \hat{A} \rangle$ in the Heisenberg equations.

The field in the semiclassical approximation is a classical dynamical variable. Further simplification is obtained by treating it as an external variable. To see this, consider the Heisenberg equation

$$\mathrm{i}\dot{\hat{a}}_k = \omega_k \hat{a}_k + \sum_{i,j=1}^{N} g_{ijk}^* \hat{A}_{ij} \quad (7.12)$$

where 'dot' over a quantity denotes derivative with respect to time. The formal solution of (7.12) reads

$$\hat{a}_k(t) = \exp(-\mathrm{i}\omega_k t) \left[\hat{a}_k(0) - \mathrm{i} \int_0^t \mathrm{d}\tau \exp(\mathrm{i}\omega_k \tau) \sum_{i,j=1}^{N} g_{ijk}^* \hat{A}_{ij}(\tau) \right]. \quad (7.13)$$

First term on the right hand side of this equation describes contribution due to free evolution of the field. The second term is the reaction on the field from atomic transitions induced by it. If the applied field is sufficiently intense compared with what can be contributed by the atomic transitions then the second term in (7.13) can be ignored so that

$$\hat{a}_k(t) \approx \exp(-\mathrm{i}\omega_k t)\, \hat{a}_k(0). \quad (7.14)$$

The field then is no longer a dynamical variable. The field operators at time t may then be replaced by their averages in the initial state. We assume the field to be in the coherent state $|\{\alpha_k\}\rangle$ ($|\alpha_k| \gg 1$), replace the operator $\hat{a}_k(0)$ in (7.14) by α_k and substitute it in the hamiltonian to reduce it to

$$\hat{H} = \sum_{i=1}^{N} E_i \hat{A}_{ii} + \hbar \sum_k \hat{A}_{ij} \left[g_{ijk} \exp(-\mathrm{i}\omega_k t) \alpha_k + \mathrm{c.c.} \right]. \quad (7.15)$$

The atomic operators are now the only dynamical variables.

The solution of the dynamical problem associated with even the semiclassical hamiltonians is generally a formidable task. However, there are numerous realistic situations in which the atom-field interaction can be approximated by manageable forms. The approximations central to quantum optics are the rotating wave and the resonance approximations, discussed next.

7.2 Rotating Wave and Resonance Approximations

To understand the conceptual basis of these approximation, consider a simplified version of the hamiltonian (7.15) in which the field is assumed to be prescribed externally. Assume that it is a single mode field of frequency ω and that it couples only the levels $|g\rangle$ and $|e\rangle$ ($E_e > E_g$). The interaction part in (7.15) may be written, assuming α and g to be real, as

$$\hat{H}_{\rm int}(t) = 2g\hbar\alpha[\hat{S}_+ + \hat{S}_-]\cos(\omega t), \tag{7.16}$$

$$\hat{S}_+ = |e\rangle\langle g|, \quad \hat{S}_- = |g\rangle\langle e|, \quad \hat{S}_z = \frac{1}{2}\Big(|e\rangle\langle e| - |g\rangle\langle g|\Big). \tag{7.17}$$

The completeness relation for the two states reads

$$|g\rangle\langle g| + |e\rangle\langle e| = 1. \tag{7.18}$$

The operators $\hat{S}_\mu$ are the same as the spin-1/2 operators introduced in Sect. 1.6. On using the definition of $\hat{S}_z$ along with the completeness relation (7.18), and on shifting the zero of energy to $(E_e + E_g)/2$, the free-atom Hamiltonian (7.4) may be rewritten as

$$\hat{H}_{\rm a} = \hbar\omega_0\hat{S}_z, \quad \omega_0 = (E_e - E_g)/\hbar. \tag{7.19}$$

The hamiltonian of a two-level atom in a monochromatic external field thus reads

$$\begin{aligned}\hat{H}(t) &= \hbar\omega_0\hat{S}_z + 2\hbar g\alpha\Big[\hat{S}_+ + \hat{S}_-\Big]\cos(\omega t)\\ &= \hbar\omega\hat{S}_z + \hbar\delta\hat{S}_z + 2\hbar g\alpha\Big[\hat{S}_+ + \hat{S}_-\Big]\cos(\omega t),\end{aligned} \tag{7.20}$$

$\delta = \omega_0 - \omega$ being the *detuning* between the atomic and the field frequencies. The state $|\psi(t)\rangle$ of the atom under this hamiltonian evolves according to

$$|\psi(t)\rangle = \overleftarrow{T}\exp\left[-\frac{\rm i}{\hbar}\int_0^t {\rm d}\tau\hat{H}(\tau)\right]|\psi(0)\rangle. \tag{7.21}$$

Apply (1.51), with $\hat{A}$ in it identified as $-{\rm i}\omega\hat{S}_z$, to rewrite (7.21) as

$$|\psi(t)\rangle = \exp\left(-{\rm i}\omega t\hat{S}_z\right)\overleftarrow{T}\exp\left[-\frac{\rm i}{\hbar}\int_0^t {\rm d}\tau\hat{H}_I(\tau)\right]|\psi(0)\rangle, \tag{7.22}$$

$$\hat{H}_I(t) = \exp\left(\mathrm{i}\omega t\hat{S}_z\right)\left[\hat{H} - \hbar\omega\hat{S}_z\right]\exp\left(-\mathrm{i}\omega t\hat{S}_z\right)$$
$$= \hat{H}_{\mathrm{rwa}} + \hbar g\alpha\left[\exp(2\mathrm{i}\omega t)\hat{S}_+ + \exp(-2\mathrm{i}\omega t)\hat{S}_-\right]. \tag{7.23}$$

The second line above is obtained by using (2.44), and

$$\hat{H}_{\mathrm{rwa}} = \hbar\left[\delta\hat{S}_z + g\alpha\left(\hat{S}_+ + \hat{S}_-\right)\right]. \tag{7.24}$$

On applying (1.51) again, with $\hat{A} = -\mathrm{i}\hat{H}_{\mathrm{rwa}}/\hbar$, (7.22) assumes the form

$$|\psi(t)\rangle = \exp\left(-\mathrm{i}\omega t\hat{S}_z\right)\exp\left(-\frac{\mathrm{i}t}{\hbar}\hat{H}_{\mathrm{rwa}}\right)$$
$$\times\overleftarrow{T}\exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^t \mathrm{d}\tau\hat{H}_{II}(\tau)\right]|\psi(0)\rangle, \tag{7.25}$$

$$\hat{H}_{II}(t) = \hbar g\alpha\Big[\exp(2\mathrm{i}\omega t)\hat{S}_+(t) + \exp(-2\mathrm{i}\omega t)\hat{S}_-(t)\Big], \tag{7.26}$$

$$\hat{S}_\mu(t) = \exp\left(\frac{\mathrm{i}t}{\hbar}\hat{H}_{\mathrm{rwa}}\right)\hat{S}_\mu\exp\left(-\frac{\mathrm{i}t}{\hbar}\hat{H}_{\mathrm{rwa}}\right). \tag{7.27}$$

The $\hat{S}_\mu(t)$ may be evaluated as in Sect. 12.2 by transforming to new set of spin operators $\hat{R}_\nu$ related with $\hat{S}_\mu$ by (12.59a). In the new representation, $\hat{H}_{\mathrm{rwa}} = \hbar\Omega\hat{R}_z$, where

$$\Omega = \sqrt{4g^2\alpha^2 + \delta^2}. \tag{7.28}$$

Recalling the results of Sect. 12.2 we note that

$$\hat{S}_+(t) = \frac{2g\alpha}{\Omega}\hat{R}_z + \frac{\mathrm{i}}{2}\left(1 + \frac{\delta}{\Omega}\right)\exp\left(\mathrm{i}\Omega t\right)\hat{R}_+$$
$$+\frac{\mathrm{i}}{2}\left(1 - \frac{\delta}{\Omega}\right)\exp\left(-\mathrm{i}\Omega t\right)\hat{R}_-\Bigg\}, \tag{7.29}$$

$$\hat{S}_z(t) = \frac{\delta}{\Omega}\hat{R}_z - \frac{\mathrm{i}g\alpha}{\Omega}\left(\exp\left(\mathrm{i}\Omega\ t\right)\hat{R}_+ - \mathrm{h.c.}\right). \tag{7.30}$$

Combination of these results in (7.26) determines $\hat{H}_{II}(t)$.

The time-ordered integral (7.25), read with (7.26) and (7.29) cannot be performed analytically exactly even for $\delta = 0$. Its approximate value of practical interest is obtained by examining the expansion (1.43) of a time-ordered integral. It shows that each time integration contributes the factors $1/\omega$, $1/(2\omega \pm \Omega)$. Hence, (7.25) is expressible as an expansion in powers of $g\alpha/\omega$, $g\alpha/(2\omega \pm \Omega)$ and their products. We consider transitions between bound state far below the ionisation limit. The strength of optical fields inducing such transitions and the atomic dipole moments are such that

$$\Omega \ll \omega. \tag{7.31}$$

Note from (7.30) that Ω is the frequency with which the atomic popualation is exchanged between the two levels. It is known as the *Rabi frequency.* The condition (7.31) states that the Rabi frequency be much smaller than the frequency of the field. For a simple argument revealing that $|g||\alpha|$ of the order of optical frequencies would ionize an atom, see [79]. Now, due to (7.31), the terms containing $g\alpha/\omega$ and $g\alpha/(2\omega + \Omega)$ are negligibly small. The terms proportional to $g\alpha/(2\omega - \Omega)$ are also small except when $\omega \approx \omega_0/3$ in which case $g\alpha/(2\omega - \Omega) \sim \omega/g\alpha$ which is large. However, an inspection of (7.29) reveals that the coefficient of $\exp(\pm 2i(\omega - \Omega)t)$, which is responsible for such terms, contributes additional factor which is of the order of $g^2\alpha^2/\omega^2$. Hence the overall contribution even from the terms like $g\alpha/(\omega - \Omega)$ is of the order of $g\alpha/\omega$.

In view of the preceeding deliberations, we carry the time-ordered integration in (7.25) to second-order. Now, the time scales of observation are orders of magnitude longer than ω^{-1}. Hence, what is observed is an average over many cycles of oscillations at frequency ω. Consequently, the contribution from a term oscillating at frequency of the order of ω averages to zero. Bearing this in mind, we find that non-zero contribution in second order of perturbation comes only from the terms containing $\exp(\pm it\Omega)$. It follows that

$$\overleftarrow{T} \exp\left(-\frac{\mathrm{i}}{\hbar}\int_0^t \mathrm{d}\tau \hat{H}_{II}(\tau)\right) \approx 1 - 2\mathrm{i}\frac{g^3\alpha^3 t}{\omega\Omega}\hat{S}_z \approx \exp\left(-\mathrm{i}\delta_{BS} t\hat{S}_z\right), \quad (7.32)$$

where, for the reasons elaborated below,

$$\delta_{BS} = \frac{2g^3\alpha^3}{\omega\Omega} \quad (7.33)$$

describes shift in the atomic frequency ω_0. It is known as the *Bloch–Siegert shift.* The expression (7.32) may alternatively be derived by evaluating (1.108) to second order. We substitute (7.32) in (7.25) and invoke the smallness of $g\alpha/\omega$ to write the product of the exponential containing $\hat{H}_{\mathrm{rwa}}$ and (7.32) as $\exp\{-\mathrm{i}t(\hat{H}_{\mathrm{rwa}}/\hbar + \delta_{BS}\hat{S}_z)\}$. The expression (7.25) then reduces to

$$\begin{aligned}|\psi(t)\rangle &= \exp\left(-\mathrm{i}\omega t\hat{S}_z\right)\exp\{-\mathrm{i}t(\hat{H}_{\mathrm{rwa}}/\hbar + \delta_{BS}\hat{S}_z)\}|\psi(0)\rangle \\ &= \overleftarrow{T}\exp\left[-\mathrm{i}\int_0^t \mathrm{d}\tau \hat{H}(\tau)/\hbar\right]|\psi(0)\rangle, \end{aligned} \quad (7.34)$$

$$\hat{H} = \hbar(\omega_0 + \delta_{BS})\hat{S}_z + \hbar g\alpha\left(\exp(-\mathrm{i}\omega t)\hat{S}_+ + \exp(\mathrm{i}\omega t)\hat{S}_-\right). \quad (7.35)$$

The reader may verify that the second line in (7.34) reduces to the first if (1.51) is applied with $\hat{A} = -\mathrm{i}\omega\hat{S}_z$. The hamiltonian (7.35) shows that in the process of making approximations, the atomic transition frequency is shifted by δ_{BS}. This shift is negligibly small for optical frequencies. In what follows we ignore δ_{BS}. We note that the terms ignored in the approximate form (7.35) of (7.16) are $\exp(\mathrm{i}\omega t)\hat{S}_+$ and $\exp(-\mathrm{i}\omega t)\hat{S}_-$. It is known as the *rotating*

wave approximation (RWA). Dropped terms are the so called *counter rotating* terms. For an explanation of this nomenclature, see, for example [79].

The interaction hamiltonian is simplified still further if the detuning between the field and the atomic transition frequency is large. To that end, note that an atomic operator $\hat{S}_\mu$, under the action of (7.35), evolves to

$$\begin{aligned}\hat{S}_\mu(t) &= \overleftarrow{T} \exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^t \hat{H}(\tau)\mathrm{d}\tau\right] \hat{S}_\mu \overrightarrow{T} \exp\left[\frac{\mathrm{i}}{\hbar}\int_0^t \hat{H}(\tau)\mathrm{d}\tau\right] \\ &= \exp(-\mathrm{i}\hat{S}_z\omega t)\exp\left[-\frac{\mathrm{i}}{\hbar}\hat{H}_{\mathrm{rwa}}t\right]\hat{S}_\mu \exp\left[\frac{\mathrm{i}}{\hbar}\hat{H}_{\mathrm{rwa}}t\right] \\ &\quad \times \exp(\mathrm{i}\hat{S}_z\omega t). \end{aligned} \tag{7.36}$$

The exact analytic expressions for $\hat{S}_\mu$ under the action of $\hat{H}_{\mathrm{rwa}}$ have already been evaluated in (7.29). We examine them under the condition

$$|\delta| \gg g\alpha. \tag{7.37}$$

This states that on-resonance Rabi frequency be much smaller than the detuning between the atom and the field. A perturbative expansion of the exact $\hat{S}_\mu(t)$ shows that, to zeroth order in $g\alpha/\delta$, $\hat{S}_z(t) \approx \hat{S}_z(0)$. Hence, if the atom is initially in its ground state then the expectation value of $\hat{S}_z(t)$ is $-1/2$ to zeroth order in $g\alpha/\delta$. To this order, we may let $\hat{S}_z(t) \approx -1/2$. The commutation relation $[\hat{S}_+,\ \hat{S}_-] = 2\hat{S}_z$ then reduces to

$$[\hat{S}_-,\ \hat{S}_+] \approx 1. \tag{7.38}$$

This suggests that the atomic operators may be approximated by the harmonic oscillator operators:

$$\hat{S}_- \to \hat{b}, \quad \hat{S}_+ \to \hat{b}^\dagger, \quad [\hat{b}, \hat{b}^\dagger] = 1. \tag{7.39}$$

The foregoing considerations apply also when the field is treated quantum mechanically. The atom-field system in this case, in terms of spin operators introduced above, is described by the hamiltonian

$$\hat{H} = \hbar\omega_0\hat{S}_z + \hbar\omega\hat{a}^\dagger\hat{a} + \hbar\left(\hat{S}_+ + \hat{S}_-\right)\left(g\hat{a} + g^*\hat{a}^\dagger\right). \tag{7.40}$$

By following the method outlined above and by using the solution (11.22) for a two-level atom in a single-mode quantized field we find that, if

$$|g|\sqrt{\langle\hat{a}^\dagger\hat{a}\rangle} \ll \omega, \tag{7.41}$$

then (7.40) reduces to (ignoring the Bloch–Siegert shift)

$$\hat{H} = \hbar\omega_0\hat{S}_z + \hbar\omega\hat{a}^\dagger\hat{a} + \hbar\left(g\hat{S}_+\hat{a} + g^*\hat{a}^\dagger\hat{S}_-\right). \tag{7.42}$$

This is known as the *Jaynes–Cummings model* [80].

Reducing (7.40) to (7.42) constitutes the RWA for a quantized field. In this form, we see that RWA amounts to ignoring energy non-conserving terms

in which emission (absorption) of a photon is accompanied by the transition of the atom from its lower (upper) to its upper (lower) state. In other words, RWA ignores terms in which the atomic raising (lowering) operator multiplies the field creation (annihilation) operator. Similarly, following the method outlined above for a classical field, it may be shown that if

$$|\delta| \gg |g|\sqrt{\langle \hat{a}^\dagger \hat{a} \rangle} \tag{7.43}$$

then the atomic evolution is adequately described by harmonic oscillator approximation of two-level operators.

A pair of levels for which (7.43) holds are said to be *far off-resonant* with the field mode in question. Else they are said to be *nearly-resonant* with it. The deliberations above show that if a level is coupled far off-resonantly with one and nearly resonantly with another then the off-resonant coupling can be discarded in comparison with the nearly resonant one. We refer to it as the *resonance approximation.*

In what follows we assume the RWA to hold to rewrite (7.11) as

$$\hat{H} = \sum_{i=1} E_i^{(m)} \hat{A}_{ii} + \sum_k \omega_k \hat{a}_k^\dagger \hat{a}_k + \sum_k \sum_{E_i<E_j} \left[g_{ijk} \hat{a}_k \hat{A}_{ji} + \text{h.c.} \right]. \tag{7.44}$$

Based on the resonance approximation, we obtain its simplified forms for some situations of practical interest. The issue of their solution is addressed in Chap. 11.

7.3 Two-Level Atom

Consider an atom in a states $|g\rangle$ irradiated by a multichromatic field. Let there be a frequency ω in the field for which an allowed transition from $|g\rangle$ to another level $|e\rangle$ is nearly resonant. Let this transition be far off-resonant with all the other frequencies present in the field. Also, let all the allowed transitions from $|e\rangle$ or from $|g\rangle$ to any other level be far off-resonant with all the frequencies present in the field. As discussed above, the atom then interacts dominantly only with the mode of frequency ω and undergoes transitions only between $|g\rangle$ and $|e\rangle$. The atom can then be considered as a *two-level atom* and the field as monochromatic of frequency ω. The corresponding hamiltonian in RWA is obtained from (7.44) by retaining in it the free-field term corresponding only to the mode of frequency ω, the free-atom term corresponding only to the two levels and the interaction term only between those levels and the single mode. It is the Jaynes-Cummings hamiltonian given by (7.42). Its generalization to N identical two-level atoms is described by

$$\hat{H} = \hbar\omega_0 \sum_{i=1}^{N} \hat{S}_z^{(i)} + \hbar\omega \hat{a}^\dagger \hat{a} + \hbar \sum_{i=1}^{N} \left[g_i \hat{S}_+^{(i)} \hat{a} + g_i^* \hat{a}^\dagger \hat{S}_-^{(i)} \right], \tag{7.45}$$

where $S_{\pm}^{(i)}, S_z^{(i)}$ are the spin operators for the ith atom and g_i is its coupling constant. The dependence of the coupling constant on the position of the atom arises owing to the variation of the field in space. If it is assumed that that variation is negligible over the region containing the atoms then $g_i \approx g$ for all i. It reduces (7.45) to the form

$$\hat{H} = \hbar\omega_0 \hat{S}_z + \hbar\omega \hat{a}^\dagger \hat{a} + [g\hat{S}_+ \hat{a} + g^* \hat{a}^\dagger \hat{S}_-], \tag{7.46}$$

where the collective spin operators are as in (3.73).

7.4 Three-Level Atom

Consider now the situation in which only a frequency ω_a of a multichromatic field is nearly resonant with a pair of levels $|g\rangle$ and $|i\rangle$. Let the frequency ω_b be the only one nearly resonant with the pair of levels $|i\rangle$ and $|e\rangle$. If the states have definite parity then a direct transition between $|g\rangle$ and $|e\rangle$ is dipole forbidden. Let all transitions from any of the three levels in question to any other level be far off-resonance with any frequency in the field. The atom can then be visualized as having three levels and the field as having only two frequencies if $\omega_a \neq \omega_b$. It acts as a monochromatic field if $\omega_a = \omega_b$.

Note that, there are three different configurations possible for transitions between the three levels depending upon the position of the intermediate level $|i\rangle$. Those are shown in Fig. 7.1 along with the allowed transitions. In the configuration of the first diagrams in Fig. 7.1, the energy of $|i\rangle$ is intermediate between the energies of the other two levels. It is called the *ladder* configuration. The energy of $|i\rangle$ in the Λ or *Raman* configuration is higher than that of the other two levels (second diagram in Fig. 7.1) whereas the level $|i\rangle$ in the V-system is below the other two levels (third diagram in Fig. 7.1). If $(\hat{a},\ \hat{a}^\dagger)$ and $(\hat{b},\ \hat{b}^\dagger)$ are the e.m. field mode operators corresponding to the

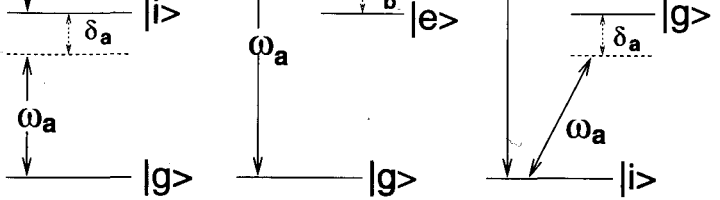

Fig. 7.1. Three configurations of a three level atom in a two-mode field each acting on only one transition.

frequencies ω_a and ω_b respectively then it is straightforward to see that the hamiltonians for the ladder, Raman and V configurations in the RWA are given respectively by

$$\hat{H}_{\rm L} = \hat{H}_0 + \hbar\left[g_a^*\hat{a}^\dagger\hat{A}_{gi} + g_a\hat{A}_{ig}\hat{a}\right] + \hbar\left[g_b^*\hat{b}^\dagger\hat{A}_{ie} + g_b\hat{A}_{ei}\hat{b}\right], \tag{7.47}$$

$$\hat{H}_{\rm R} = \hat{H}_0 + \hbar\left[g_a^*\hat{a}^\dagger\hat{A}_{gi} + g_a\hat{A}_{ig}\hat{a}\right] + \hbar\left[g_b^*\hat{b}^\dagger\hat{A}_{ei} + g_b\hat{A}_{ie}\hat{b}\right], \tag{7.48}$$

$$\hat{H}_{\rm V} = \hat{H}_0 + \hbar\left[g_a^*\hat{a}^\dagger\hat{A}_{ig} + g_a\hat{A}_{gi}\hat{a}\right] + \hbar\left[g_b^*\hat{b}^\dagger\hat{A}_{ie} + g_b\hat{A}_{ei}\hat{b}\right]. \tag{7.49}$$

In the equations above, $\hat{H}_0 = \hat{H}_{\rm f} + \hat{H}_{\rm a}$ where

$$\hat{H}_{\rm f} = \hbar\omega_a\hat{a}^\dagger\hat{a} + \hbar\omega_b\hat{b}^\dagger\hat{b}, \quad \hat{H}_{\rm a} = \sum_{m=g,i,e} E_m\hat{A}_{mm} \tag{7.50}$$

are the free field and the free atom hamiltonians. The problem of solving these hamiltonians is addressed in Chap. 11.

7.5 Effective Two-Level Atom

In the processes described above, a photon mediates transitions between a pair of levels coupled by an allowed dipole transition. This is the lowest order process in the interaction of radiation with atoms. If there is no pair of direct dipole-coupled levels available close to an applied frequency, higher order processes come in to play. The process next in line is a *two-photon process.* As the name suggests, it describes a situation in which transition between two levels is mediated by simultaneous action of two photons. The action in question may consist in simultaneous absorption or emission of two photons, or absorption of one accompnaied by simultaneous emission of the other. The former process takes place if the sum of two applied frequencies is close to the frequency of transition between a pair of levels having a non zero dipole matrix element for the said transition. The latter process applies when the difference of two applied frequencies is close to the frequency of transition between a pair of levels having a non zero dipole matrix element for the said transition. For the sake of definiteness, let $\hat{a}$ and $\hat{b}$ be the field annihilation operators for the modes of frequencies ω_a and ω_b. If the levels $|e\rangle$ and $|g\rangle$ are such that $\hbar\omega_0 \equiv E_{\rm e} - E_{\rm g} \approx \hbar(\omega_a + \omega_b)$ then the interaction between the field and the atom is goverened by the hamiltonians

$$\hat{H}_{\rm L} = \hbar\left[d_{\rm eg}^{(2)} e_a e_b \hat{a}\hat{b}|e\rangle\langle g| + \text{h.c.}\right]. \tag{7.51}$$

Here, $d_{\rm eg}^{(2)}$ is the dipole matrix element for the process in question and e_a, e_b are as in (7.10).

In case $E_e - E_g \approx \hbar(\omega_a - \omega_b)$ then

$$\hat{H}_R = \hbar \left[d_{eg}^{(1,1)} e_a e_b^* \hat{a}\hat{b}^\dagger |e\rangle\langle g| + \text{h.c.} \right], \tag{7.52}$$

the $d_{eg}^{(1,1)}$ being the dipole matrix element for the process of absorption of one and emission of another photon. For further details see [81]–[83].

These considerations can be generalized to a multiphoton process involving m emissions and n absorptions. Such a process is described by the hamiltonian of the form (7.57).

The strength of a multiphoton process is enhanced if there is an intermediate transition close to a field frequency participating in the multiphoton process but still sufficiently far away enabling that transitions to be termed far-off resonant. We elaborate this in the following.

To that end, in the three-level models introduced in the last section, let the detunings

$$\delta_a = \frac{|E_i - E_g|}{\hbar} - \omega_a, \quad \delta_b = \frac{|E_e - E_i|}{\hbar} - \omega_b \tag{7.53}$$

of each pair of levels from the frequency of their respective coupling fields be such that $|g_a|\sqrt{\bar{n}_a} \ll |\delta_a|$, $|g_b|\sqrt{\bar{n}_b} \ll \delta_b$. Here $\bar{n}_a, \bar{n}_b$ are the average number of photons of frequency ω_a and ω_b. In this case, the transition between $|g\rangle$ and $|i\rangle$ as well as that between $|e\rangle$ and $|i\rangle$ are far off-resonant. We can then solve the Heisenberg equations perturbatively with g_a/δ_{ig} and g_b/δ_{ie} as the perturbation parameters. However, exact solutions for the three configurations of a three-level atom are also available (see Chap. 11) under the condition

$$E_e - E_g = \hbar(\omega_a + \omega_b), \quad \text{ladder configuration,} \tag{7.54a}$$

$$E_e - E_g = \hbar|\omega_a - \omega_b|, \quad \text{Raman and V configurations.} \tag{7.54b}$$

Starting with the exact solution we have carried the said perturbation expansion in Chap. 11. We find that, to the lowest order of perturbation, the intermediate level remains unoccupied if it is unoccupied initially. The three-level atom is thus reduced to an *effective two-level atom.* The dynamics of the three configurations is found to be governed by the following effective two-level hamiltonians

$$\hat{H}_L = \hat{H}_f + \left(E_g + \frac{|g_a|^2}{\delta} \hat{a}^\dagger \hat{a} \right) \hat{A}_{gg} + \left(E_e + \frac{|gb|^2}{\delta} \hat{b}\hat{b}^\dagger \right) \hat{A}_{ee} + \left[\frac{g_a g_b}{\delta} \hat{a}\hat{b}\hat{A}_{eg} + \text{h.c.} \right], \tag{7.55}$$

$$\hat{H}_R = \hat{H}_f + \left(E_g + \frac{|g|_a^2}{\delta} \hat{a}^\dagger \hat{a} \right) \hat{A}_{gg} + \left(E_e + \frac{|g_b|^2}{\delta} \hat{b}^\dagger \hat{b} \right) \hat{A}_{ee} + \left[\frac{g_b^* g_a}{\delta} \hat{b}^\dagger \hat{a} \hat{A}_{eg} + \text{h.c.} \right], \tag{7.56}$$

The hamiltonian in the V-configuration is similar to that in the Raman configuration. Each term in the expressions above has a simple physical meaning:

- The operator $\hat{A}_{eg}\hat{a}\hat{b}$ in (7.55) describes the process of the atom making transition from its lower state $|g\rangle$ to its upper state $|e\rangle$ by absorbing simultaneously one photon of frequency ω_a and another of frequency ω_b. The operator $\hat{a}^\dagger\hat{b}^\dagger\hat{A}_{ge}$ describes the reverse process. In the same way, $\hat{a}\hat{b}^\dagger\hat{A}_{eg}$ in (7.56) describes transition from $|g\rangle$ to $|e\rangle$ by simultaneous absorption of a photon of frequency ω_a and emission of a photon of frequency ω_b. The strength of these processes is evidently enhanced by decreasing the detuning of the intermediate level to the extent that it still qualifies to be called far off resonance.
- The shift in the energy levels in (7.55)–(7.56) is the *Stark shift*. Note that it depends on the intensity of the field. It arises due to virtual transitions to the intermediate level. The atom in the lower state in the ladder configuration makes virtual transitions to the intermediate state by simultaneous absorption and emission of a photon of frequency ω_a. Note that in this case atom must absorb a photon first and then emit. This process is represented by $\hat{a}^\dagger\hat{a}$. The atom in this process remains in the state $|g\rangle$. It is, therefore, described by the part of the Stark shift term reading $\hat{a}^\dagger\hat{a}\hat{A}_{gg}$. The analogous process for the atom in the upper state $|e\rangle$ is, however, performed by emission followed by simultaneous absorption of a photon of mode b. This is represented by $\hat{b}\hat{b}^\dagger$. Note the order of the operators. The atom, of course, remains in the state $|e\rangle$. The process of virtual transition from the upper state is thus represented by $\hat{b}\hat{b}^\dagger\hat{A}_{ee}$. The Stark shift terms in the other configurations can be similarly interpreted.

A process in which $\omega_a \neq \omega_b$ is termed *non-degenerate*. Else it is a *degenerate process*. The hamiltonians for the degenerate processes corresponding to (7.55), (7.56) are obtained by the replacements $\omega_b \to \omega_a$, $\hat{b} \to \hat{a}$.

The interpretation of the effective two-level models outlined above provides a recipe for eliminating one or more off-resonant levels. If two levels are coupled by n off-resonant dipole transitions between them which involve absorption of m photons and emission of n in going from $|g\rangle$ to $|e\rangle$ then the hamiltonian, in the interaction picture generated by the free hamiltonian, will be of the form

$$\hat{H}_I(t) = g\exp\left(-\mathrm{i}\Delta t\right)\hat{a}_1\cdots\hat{a}_m\hat{b}_1^\dagger\cdots\hat{b}_n^\dagger|e\rangle\langle g| + \text{h.c.} \tag{7.57}$$

Here $\Delta = \omega_1 + \cdots\omega_m - \omega_{m+1} + \cdots\omega_n - \omega_0$. This describes a *coherent multiphoton process*. The problem of solution of such a hamiltonian is addressed in Chap. 11.

7.6 Multi-channel Models

The atomic levels configurations and the fields in the models above are such that only one mode couples a pair of levels. Several interesting and useful phenomena are observed when a pair of levels is coupled by two or more frequencies. A phenomenon of particular interest, possible with multi-channel models, is the *wave mixing*. It refers to the phenomenon of generation of a new frequency by algebraic addition of frequencies or by a division of a frequency. In order to see that, let the hamiltonian in the interaction picture be

$$\hat{H}_I(t) = \exp(-\mathrm{i}\Delta t)\hat{F}|e\rangle\langle g| + \exp(\mathrm{i}\Delta t)\hat{F}^\dagger|g\rangle\langle e|. \tag{7.58}$$

Notice that the combination of the field operators responsible for the transition that starts and ends up at $|g\rangle$ is $\hat{F}^\dagger\hat{F}$. Comparing (7.58) with the hamiltonian (7.57) describing a single-channel multi-photon process, we see that $\hat{F}$ for such a process is a product of annihilation and creation operators for various modes. The operator $\hat{F}^\dagger\hat{F}$ is, therefore, a product of number operators for the modes in question. Hence, at the end of a cycle, the energy of every mode is the same as at the beginning.

Exchange of energy between modes at the completion of a cycle may take place if $\hat{F}$ is a sum of products of field operators. As an example, consider the process depicted in Fig. 7.2. In it, the atom can go from level $|g\rangle$ to $|e\rangle$ by absorbing two photons, one of frequency ω_a and another of frequency ω_b. It can make that transition alternatively by absorbing a photon of frequency ω_c. Notice that parity considerations would prevent simultaneous occurrence of these two processes if the states involved are of definite parity. This process can therefore occur only if the states are not of definite parity. Since parity conservation is violated in the absence of space-inversion symmetry, for the process in question to be possible, atom must be in a space-inversion symmetry breaking environment like in a crystal. The process of Fig. 7.2 is

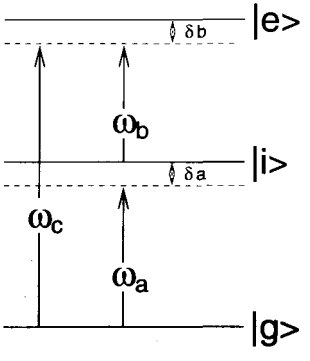

Fig. 7.2. Effective two-level atom in which levels are coupled by two channels.

described by (7.58) with

$$\hat{F} = g_1 \hat{a}\hat{b} + g_2 \hat{c}, \quad \Delta = \omega_c - \omega_0, \quad \omega_c = \omega_a + \omega_b. \tag{7.59}$$

In this case $\hat{F}^\dagger \hat{F}$ involves terms like $\hat{a}^\dagger \hat{b}^\dagger \hat{c}$ which do not commute with the number operator of any of the fields. There is thus an exchange of energy between the modes at the completion of a cycle. We discuss the process of wave mixing arising out of this hamiltonian in the next section.

As another example, consider the process depicted in Fig. 7.3. It describes generation of Stokes and anti-Stokes fields of frequency ω_s and ω_a on interaction with a pump of frequency ω_p with a three-level atom in Raman configuration in which the intermediate level is off-resonant with all the transitions. The corresponding interaction hamiltonian is (7.58) with [84]

$$\hat{F} = g_s \hat{a}_S^\dagger \hat{a}_p + g_a \hat{a}_p^\dagger \hat{a}_A. \tag{7.60}$$

This is the so called *two-channel Raman-coupled model* [84]. For some other two-channel models, see [85].

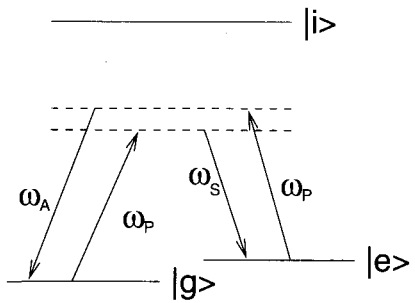

Fig. 7.3. Two-channel Raman-coupled model.

7.7 Parametric Processes

In the processes introduced above, the state of the atom as well as that of the field changes. *Parametric processes* refer to the class of processes in which an atom in a level makes only virtual transitions to other levels. Consider the interaction Hamiltonian (7.58). Let $|\Delta| \gg ||\hat{F}||$ so that the transitions between $|g\rangle$ and $|e\rangle$ are off-resonant. Let the atom be initially in state $|g\rangle$. The transitions to $|e\rangle$ are then only virtual. The procedure outlined in Sect. 7.5 leads to the following hamiltonian for the fields:

$$\hat{H}_{\mathrm{P}} = \hat{H}_{\mathrm{f}} + \frac{1}{\Delta} \hat{F}^\dagger \hat{F}. \tag{7.61}$$

As discussed in the last section, wave-mixing can occur if $\hat{F}$ is a sum of products of field operators. As an example, (7.61) for $\hat{F}$ given by (7.59) (with $g_1 = g_2, G = |g|_1^2/\Delta$) reads

$$\hat{H}_{\rm P} = \hat{H}_{\rm f} + G\left[\hat{c}^\dagger \hat{a}\hat{b} + \hat{b}^\dagger \hat{a}^\dagger \hat{c}\right]. \tag{7.62}$$

Now, we assume that the mode c is initially in an intense coherent state $|\alpha\rangle$, $|\alpha| \gg 1$. On account of the procedure outlined in Sect. 7.1, we treat the mode c classically by replacing $\hat{c}$ by $\alpha \exp(-\mathrm{i}\omega_c t)$ so that (7.62) reduces to

$$\hat{H}_{\rm P} = \omega_a \hat{a}^\dagger \hat{a} + \omega_b \hat{b}^\dagger \hat{b} + \left[\xi^* \hat{a}\hat{b} \exp(\mathrm{i}\omega_c t) + \text{h.c.}\right]. \tag{7.63}$$

$\xi = G\alpha$. The dynamics generated by this Hamiltonian may be investigated by recalling from (2.71) that it is an element of $SU(1,1)$. An application of the similarity transformation (2.75) shows that the modes a and b grow in time. The Hamiltonian (7.63) thus transfers energy of the mode c, called the *pump* to the modes a and b. The Hamiltonian (7.63) thus describes the process of *down conversion* of a frequency.

7.8 Cavity QED

It should be clear that the models introduced above, and their likes, rely on the availability of a set of well-separated discrete modes. This may be achieved only inside an appropriately constructed cavity. Recall that the possible values of wavelength inside a cavity of finite volume V are discretely spaced and that the density of modes at frequency ν is proportional to $(l^2 + m^2 + n^2)$ where l, m, n are integers characterizing the order of a mode. This implies that lower order modes are better suited to meet the requirement of well-separation of frequencies. For cavities of the size of a few millimeters, lower order frequencies fall in the microwave region.

However, the field in a cavity loses energy due to leakage at the walls of the cavity. The methods for accounting for such losses are outlined in Chap. 8. The models introduced above ignore such loses. Hence, in order to realize these models experimentally, the time of interaction $t_{\rm int}$ should be such that $t_{\rm int} \ll \kappa^{-1}$ where κ is the rate of leakage of the field at frequency ω. That rate is usually expressed as $\kappa = \omega/Q$ where Q is called the cavity Q-factor.

We consider a microwave cavity with high enough Q factor and find out what kind of atoms are suitable for realizing the said models. To that end, note that in order to enable the atoms to pass through it, the cavity has to be open ended. The modes travelling parallel to the axis of the cavity are, of course, discretely spaced. However, the modes entering from the open sides form a continuum. We will see in Chap. 8 that interaction with a continuum of modes causes an atom to decay spontaneously. It is obviously an undesirable hurdle in the way of realization of the models in question. We, therefore, need to restrict the time of interaction to a time scale much shorter than the time

scale of spontaneous decay. The time of interaction, however, should be long enough to allow for appreciable exchange of energy between the atom and the field. Recall that the rate of exchange of energy between a pair of atomic levels coupled resonantly with a field mode is given by the Rabi frequency which is proportional to the dipole moment of the transition in question. Hence, the dipole moment of the operative transition should be high enough to make its Rabi frequency g much larger than the spontaneous decay rate γ from any of the operative levels. These considerations imply that the experimental conditions should be such that

$$g^{-1} \ll t_{\text{int}} \ll \gamma^{-1}, \kappa^{-1}. \tag{7.64}$$

For experimentally meaningful interaction times in microwave cavities, such a condition can be realized with transitions between states of high principal quantum number n called the *Rydberg states*. An atom in such a state is called a Rydberg atom. In the following we enumerate some properties of Rydberg atoms to bring out their usefulness for the purpose in question.

1. The energy of a level of principal quantum number n is proportional to n^{-2}. Hence, the frequency ω of transition between the levels of neighbouring principal quantum number $\sim n^{-3}$ if n is high. For $n \sim 30$, ω lies in the microwave range.
2. The dipole moment d for transitions between nearby levels is proportional to n^2. Clearly, the dipole moment for $n \sim 30$ is almost two orders of magnitude higher than normal optical transitions corresponding to, say, $n \sim 3$. The choice of high n is thus in consonance with the aforementioned requirement of strong atom-field coupling.
3. We will see in Chap. 8 that the rate of spontaneous emission γ between levels having dipole matrix element d and transition frequency ω is proportional to $d^2\omega^3$. Hence, in view of the properties 1 and 2 above, $\gamma \sim n^{-5}$ for transition between two Rydberg levels. The rate of spontaneous decay between two Rydberg levels is thus reduced considerably compared with that in optical domain. For $n \sim 30$, $\gamma \sim 100\ \text{sec}^{-1}$.
4. The spontaneous emission in 3 above is for transitions between two operative Rydberg levels. An atom may decay spontaneously to other lower energy levels. The total rate Γ of such an emission may be shown to be $\sim n^{-3}$ [86]. However, emission to states other than the ones in question does not have any bearing on the model since the probability of an atom returning to an operative level from a non-operative one is negligible.

These considerations show that the conditions appropriate to realizing the models in question can be met by working with Rydberg atoms on a time scale of few milliseconds.

The problem of interaction of isolated atomic transitions with isolated e.m. field modes is thus a problem of cavity quantum electrodynamics (QED). See [86, 87] for details of applications of cavity QED in probing fundamental aspects of quantum mechanics and atom-field interaction.

7.9 Moving Atom

In our treatment so far of atom-field interaction, we have tacitly assumed that the atom is fixed at one position or, if the atom is moving, there is no spatial variation of the field over the distance that it traverses during the duration of its interaction with it. These conditions exclude many important situations of interest. For example, in cavity QED, an atom enters a cavity and interacts with a spatially varying field while in motion. An atom oscillating in a laser cooled trap is another example which is of current interest.

The atomic motion may be included in the description of atom-field interaction by making the position vector $\boldsymbol{r}$ in the mode function in the interaction hamiltonian a dynamical variable so that

$$\hat{\tilde{H}} = \frac{1}{2m}\hat{\boldsymbol{P}}^2 + V(\hat{\boldsymbol{r}}) + \hat{H}_{\rm a} + \hat{H}_{\rm f} + \hat{H}_{\rm a-f}(\hat{\boldsymbol{r}}). \tag{7.65}$$

Here $\boldsymbol{P}$ is the momentum conjugate to $\boldsymbol{r}$, m is the atomic mass, $V(\boldsymbol{r})$ is the external potential and $\hat{H}_{\rm a-f}(\boldsymbol{r})$ is the atom-field interaction hamiltonian. Now, the momentum of an atom changes due to the influence of $V(\boldsymbol{r})$ and due to the fact that it recoils to compensate for the momentum of radiation emitted or absorbed by it. Each process of emission or absorption of radiation of wave vector $\boldsymbol{k}$ changes the atomic momentum by $\hbar|\boldsymbol{k}|$. If this is negligibly small compared with the momentum of the atom then we can ignore it and assume that the atomic momentum changes solely due to $V(\boldsymbol{r})$. Hence, if $\boldsymbol{r}_m(t)$ is the position vector at time t due to the evolution under the mechanical part $\hat{\boldsymbol{P}}^2/2m + V(\boldsymbol{r})$ of the hamiltonian then the effect of atomic motion on atom-field interaction is adequately described by replacing $\hat{\boldsymbol{r}}$ in $\hat{H}(\hat{\boldsymbol{r}})$ by $\langle \boldsymbol{r}_m(t)\rangle$ so that

$$\hat{\tilde{H}} = \hat{H}_{\rm a} + \hat{H}_{\rm f} + \hat{H}_{\rm a-f}(\langle \hat{\boldsymbol{r}}_m(t)\rangle). \tag{7.66}$$

The position is thus no longer a dynamic variable. Many a times it is adequate to describe the mechanical part classically. In this case, and for an atom evolving freely ($V = 0$) with fixed momentum $\boldsymbol{p}$, $\langle \hat{\boldsymbol{r}}_m(t)\rangle \to \boldsymbol{r}_0 + \boldsymbol{p}t/m$.

A system of current interest in which the atomic motion is coupled to its electronic transitions is that of an atom in a laser cooled trap. The motion in the trap is well approximated as harmonic. For a detailed review of the quantum optics of an atom in a laser cooled harmonic trap, see [88].

8. Quantum Theory of Damping

The evolutions governed by the hamiltonians we have introduced are reversible in time. However, irrevesible motions are facts of life. Spontaneous emission from an excited atomic level, absorption of radiation etc. are some examples of interest to us. How irreversibility arises is the issue addressed in this chapter. We will see that irreversible evolution is generally an outcome of a system's interaction with the environment having nondenumerably infinite number of degrees of freedom. Such an evolution is described by a so called master equation. We derive the master equation under the conditions frequently encountered in quantum optics and specialize it to various model systems. For further reading, see [89]–[91].

8.1 The Master Equation

Consider an isolated system composed of subsystems named S and R. We are interested in the dynamical behaviour of the system S alone. In what follows we refer to S as the 'system' and R as the 'reservoir'. Let the hamiltonian of the combined system $S+R$ be expressible as

$$\hat{H} = \hat{H}_{\rm S} + \hat{H}_{\rm R} + \hat{H}_{\rm RS} \equiv \hat{H}_0 + \hat{H}_{\rm RS}. \tag{8.1}$$

Here, $\hat{H}_{\rm S}$ and $\hat{H}_{\rm R}$ describe free evolution respectively of the system and the reservoir, and $\hat{H}_{\rm RS}$ is their interaction hamiltonian. We know from (1.121) that the density operator $\hat{\rho}(t)$ of the combined system at time t is given in terms of $\hat{\rho}(0)$ by the relation

$$\hat{\rho}(t) = \exp\left[-\frac{\rm i}{\hbar}\hat{H}t\right]\hat{\rho}(0)\exp\left[\frac{\rm i}{\hbar}\hat{H}t\right] \equiv \exp\left[\hat{\hat{L}}t\right]\hat{\rho}(0), \tag{8.2}$$

where $\hat{\hat{L}}\hat{\rho} = (-{\rm i}/\hbar)[\hat{H},\ \hat{\rho}]$. On using (8.1) for $\hat{H}$ and on applying (1.51), this expression may be rewritten as

$$\begin{aligned}\hat{\rho}(t) = {} & \exp\left[-\frac{\rm i}{\hbar}\hat{H}_0 t\right]\overleftarrow{T}\exp\left[-\frac{\rm i}{\hbar}\int_0^t {\rm d}\tau\hat{H}_{\rm I}(\tau)\right]\hat{\rho}(0) \\ & \times\overrightarrow{T}\exp\left[\frac{\rm i}{\hbar}\int_0^t {\rm d}\tau\hat{H}_{\rm I}(\tau)\right]\exp\left[\frac{\rm i}{\hbar}\hat{H}_0 t\right],\end{aligned} \tag{8.3}$$

$$\hat{H}_{\rm I}(t) = \exp\left(\frac{\rm i}{\hbar}\hat{H}_0 t\right)\hat{H}_{\rm RS}\exp\left(-\frac{\rm i}{\hbar}\hat{H}_0 t\right). \tag{8.4}$$

The behaviour of S alone is determined by the density operator $\hat{\rho}_{\rm S}(t) = {\rm Tr}_{\rm R}[\hat{\rho}(t)]$ obtained by performing the operation of trace over the reservoir operators in the combined density operator $\hat{\rho}(t)$. Let S and R be decoupled at $t = 0$ so that

$$\hat{\rho}(0) = \hat{\rho}_{\rm S}(0)\otimes\hat{\rho}_{\rm R}(0), \quad {\rm Tr}_{\rm S}\hat{\rho}_{\rm S}(0) = {\rm Tr}_{\rm R}\hat{\rho}_{\rm R}(0) = 1. \tag{8.5}$$

Substitute this in (8.3) along with the definition $\hat{H}_0 = \hat{H}_{\rm S} + \hat{H}_{\rm R}$ and perform trace over R. Since $\hat{H}_S$ does not contain any reservoir operator, the exponential containing it can be brought out of the trace. The cyclic property of the trace can be applied to the reservoir operators. As a result we find that

$$\hat{\rho}_{SI}(t) = \hat{\hat{D}}(t)\hat{\rho}_{\rm S}(0), \tag{8.6}$$

$$\hat{\rho}_{\rm SI}(t) = \exp\left[\frac{\rm i}{\hbar}\hat{H}_{\rm S}t\right]\hat{\rho}_{\rm S}(t)\exp\left[-\frac{\rm i}{\hbar}\hat{H}_{\rm S}t\right], \tag{8.7}$$

$$\begin{aligned}\hat{\hat{D}}(t)\hat{\rho}_{\rm S}(0) &= {\rm Tr}_{\rm R}\left\{\overleftarrow{T}\exp\left[-\frac{\rm i}{\hbar}\int_0^t {\rm d}\tau\hat{H}_{\rm I}(\tau)\right]\hat{\rho}_{\rm R}(0)\hat{\rho}_{\rm S}(0)\right.\\ &\quad\left.\times\overrightarrow{T}\exp\left[\frac{\rm i}{\hbar}\int_0^t {\rm d}\tau\hat{H}_{\rm I}(\tau)\right]\right\}\\ &\equiv {\rm Tr}_{\rm R}\left\{\overleftarrow{T}\exp\left[-\frac{\rm i}{\hbar}\int_0^t {\rm d}\tau\hat{\hat{L}}_I(\tau)\right]\hat{\rho}_{\rm R}(0)\right\}\hat{\rho}_{\rm S}(0).\end{aligned} \tag{8.8}$$

The superoperator $\hat{\hat{L}}_I(t)$ in the equation above is defined by

$$\hat{\hat{L}}_I(t)\hat{\rho} = \left[\hat{H}_{\rm I}(t),\ \hat{\rho}\right]. \tag{8.9}$$

Express the time-ordered integral as in (1.43) to rewrite (8.8) as

$$\hat{\hat{D}}(t) = 1 + \hat{\hat{x}}, \tag{8.10}$$

$$\hat{\hat{x}} = \sum_{n=1}^{\infty}\int_0^t {\rm d}\tau_n\int_0^{\tau_n}{\rm d}\tau_{n-1}\cdots\int_0^{\tau_2}{\rm d}\tau_1{\rm Tr}_{\rm R}\left\{\hat{\hat{L}}_I(\tau_n)\cdots\hat{\hat{L}}_I(\tau_1)\hat{\rho}_{\rm R}(0)\right\}. \tag{8.11}$$

Owing to the reasons outlined circa (1.108), we reexpress $\hat{\hat{D}}(t)$ as

$$\hat{\hat{D}}(t) = \exp\left[\ln(1+\hat{\hat{x}})\right] = \exp\left[\sum_{m=1}^{\infty}\hat{\hat{M}}_k(t)\right], \tag{8.12}$$

$$\hat{\hat{M}}_1(t) = -\frac{\mathrm{i}}{\hbar}\int_0^t \mathrm{d}\tau \mathrm{Tr}_\mathrm{R}\left[\hat{\hat{L}}_I(\tau)\hat{\rho}_\mathrm{R}(0)\right],$$

$$\hat{\hat{M}}_2(t) = \left(-\frac{\mathrm{i}}{\hbar}\right)^2 \int_0^t \mathrm{d}\tau_2 \int_0^{\tau_2} \mathrm{d}\tau_1 \mathrm{Tr}_\mathrm{R}\left[\hat{\hat{L}}_I(\tau_2)\hat{\hat{L}}_I(\tau_1)\hat{\rho}_\mathrm{R}(0)\right] - \frac{1}{2}\hat{\hat{M}}_1^2(t), \tag{8.13}$$

and so on. In writing the second step in the equation above, we have (i) expanded $\ln(1+\hat{\hat{x}})$ in powers of $\hat{\hat{x}}$, and (ii) grouped together the terms having the same number of $\hat{\hat{L}}_I$. The density operator $\hat{\rho}_\mathrm{S}(t)$ of the system S may now be determined by substituting (8.12) for $\hat{\hat{D}}(t)$ in (8.6). However, exact evaluation of $\hat{\hat{D}}(t)$ is generally a formidable task. Approximate expressions may be derived by exploiting realistic conditions. A practical situation of widespread interest is the one in which the interaction hamiltonian is much weaker than the free hamiltonian. In this case, it is adequate to retain in (8.12) terms up to the second order in the interaction. It is called the *Born approximation.* Moreover, we assume that $\hat{\rho}_\mathrm{R}(0)$ is such that $\mathrm{Tr}_\mathrm{R}[\hat{H}_\mathrm{I}(t)\hat{\rho}_\mathrm{R}(0)] = 0$ so that $\hat{\hat{M}}_1 = 0$. Hence, in the Born approximation,

$$\hat{\rho}_{SI}(t) = \exp\left(\hat{\hat{M}}_2(t)\right)\hat{\rho}_S(0). \tag{8.14}$$

To derive the equation obeyed by $\hat{\rho}_{SI}(t)$, differentiate (8.14) with respect to t using the identity (2.3). In order to be consistent with the Born approximation, the terms only up to the second order in the interaction need be retained. The operation of differentiation of the exponential operator is then equivalent with that of a c-number exponential. On recalling the definition (8.13) read with (8.9) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}_{SI}(t) = -\frac{1}{\hbar^2}\int_0^t \mathrm{d}\tau \mathrm{Tr}_\mathrm{R}\left[\hat{H}_\mathrm{I}(t), \left[\hat{H}_\mathrm{I}(t-\tau),\ \rho_\mathrm{R}(0)\hat{\rho}_{SI}(t)\right]\right]. \tag{8.15}$$

Now, let the system-reservoir interaction hamiltonian be expressible as

$$\hat{H}_\mathrm{RS} = \hbar\sum_{k=1}^N \left(\hat{S}_k^\dagger \hat{F}_k + \hat{F}_k^\dagger \hat{S}_k\right), \tag{8.16}$$

$\hat{F}_k$ being a reservoir operator and $\hat{S}_k$ a system operator. Insert this in (8.4) to obtain

$$\hat{H}_\mathrm{I}(t) = \hbar\sum_{k=1}^N \left(\hat{S}_{kI}^\dagger(t)\hat{F}_{kI}(t) + \hat{F}_{kI}^\dagger(t)\hat{S}_{kI}(t)\right), \tag{8.17}$$

$$\hat{F}_{kI}(t) = \exp\left(\mathrm{i}\hat{H}_\mathrm{R}\, t/\hbar\right)\hat{F}_k \exp\left(-\mathrm{i}\hat{H}_\mathrm{R}\, t/\hbar\right). \tag{8.18}$$

We assume that the system operator $\hat{S}_k$ is such that

$$\hat{S}_{kI}(t) \equiv \exp\left(\frac{\mathrm{i}}{\hbar}\hat{H}_{\mathrm{S}}t\right)\hat{S}_k \exp\left(-\frac{\mathrm{i}}{\hbar}\hat{H}_{\mathrm{S}}t\right) = \exp\left(-\mathrm{i}\Omega_k t\right)\hat{S}_k. \tag{8.19}$$

On combining (8.17) and (8.15) and on transforming $\hat{\rho}_{SI}(t)$ back to $\hat{\rho}_{\mathrm{S}}(t)$ using (8.7) we find that (8.15) reduces to the *master equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}_{\mathrm{S}} = -\frac{\mathrm{i}}{\hbar}\left[\hat{H}_{\mathrm{S}},\ \hat{\rho}_{\mathrm{S}}\right] + \hat{\hat{L}}_{S0}\hat{\rho}_{\mathrm{S}} \ \equiv\ \hat{\hat{L}}_{\mathrm{S}}\hat{\rho}_{\mathrm{S}}, \tag{8.20}$$

$$\begin{aligned}\hat{\hat{L}}_{S0}\hat{\rho}_{\mathrm{S}} = \sum_{k,l}\Big[&\Big(\hat{S}_k^{\dagger}\hat{\rho}_{\mathrm{S}}\hat{S}_l - \hat{\rho}_{\mathrm{S}}\hat{S}_l\hat{S}_k^{\dagger}\Big)W_{lk}^{(1)} \\ &+\Big(\hat{S}_k\hat{\rho}_{\mathrm{S}}\hat{S}_l^{\dagger} - \hat{\rho}_{\mathrm{S}}\hat{S}_l^{\dagger}\hat{S}_k\Big)W_{lk}^{(2)} + \Big(\hat{S}_k^{\dagger}\hat{\rho}_{\mathrm{S}}\hat{S}_l^{\dagger} - \hat{\rho}_{\mathrm{S}}\hat{S}_l^{\dagger}\hat{S}_k^{\dagger}\Big)W_{lk}^{(3)} \\ &+\Big(\hat{S}_k\hat{\rho}_{\mathrm{S}}\hat{S}_l - \hat{\rho}_{\mathrm{S}}\hat{S}_l\hat{S}_k\Big)W_{lk}^{(4)} + \mathrm{h.c.}\Big].\end{aligned} \tag{8.21}$$

The *rates* $W(t)'s$ are related with the two-time averages of the reservoir operators by

$$\begin{aligned}W_{lk}^{(1)} &= \int_0^{\infty}\mathrm{d}\tau\exp(\mathrm{i}\Omega_l\tau)\mathrm{Tr}_{\mathrm{R}}\left[\hat{F}_{lI}^{\dagger}(t-\tau)\hat{F}_{kI}(t)\hat{\rho}_{\mathrm{R}}(0)\right]\mathrm{d}\tau, \\ W_{lk}^{(2)} &= \int_0^{\infty}\mathrm{d}\tau\exp(-\mathrm{i}\Omega_l\tau)\mathrm{Tr}_{\mathrm{R}}\left[\hat{F}_{lI}(t-\tau)\hat{F}_{kI}^{\dagger}(t)\hat{\rho}_{\mathrm{R}}(0)\right]\mathrm{d}\tau, \\ W_{lk}^{(3)} &= \int_0^{\infty}\mathrm{d}\tau\exp(-\mathrm{i}\Omega_l\tau)\mathrm{Tr}_{\mathrm{R}}\left[\hat{F}_{lI}(t-\tau)\hat{F}_{kI}(t)\hat{\rho}_{\mathrm{R}}(0)\right]\mathrm{d}\tau, \\ W_{lk}^{(4)} &= \int_0^{\infty}\mathrm{d}\tau\exp(\mathrm{i}\Omega_l\tau)\mathrm{Tr}_{\mathrm{R}}\left[\hat{F}_{lI}^{\dagger}(t-\tau)\hat{F}_{kI}^{\dagger}(t)\hat{\rho}_{\mathrm{R}}(0)\right]\mathrm{d}\tau.\end{aligned} \tag{8.22}$$

In writing the expressions above, the upper limit of time-integration is replaced by ∞ by assuming that the reservoir correlation time τ_c is very small compared with the time t of observation. This usually requires the reservoir to have non-denumerably infinite number of degrees of freedom (see Sect. 8.4). On the said time scale of observation, the system loses all the memory of its past. It is, therefore, called the *Markov approximation.* The time of observation should, however, be short compared with the time scale on which the system evolves.

The *Liouvillean* $\hat{\hat{L}}_{\mathrm{S}}$ in (8.20) describing the evolution of S consists of two parts. The part involving the commutator of $\hat{H}_{\mathrm{S}}$ describes its free evolution whereas the influence of the reservoir, contained in $\hat{\hat{L}}_{S0}$, describes irreversible motion.

Now, let the subsystem S, while interacting with the reservoir, be driven also by an external field whose action on the system is described by $\hat{H}_{\mathbf{ext}}$ so that total hamiltonian of the combined system is

$$\hat{H} = \hat{H}_{\mathrm{S}} + \hat{H}_{\mathbf{ext}}(t) + \hat{H}_{\mathrm{R}} + \hat{H}_{\mathrm{RS}}. \tag{8.23}$$

Let us define

$$\hat{\rho}_{\rm SI}(t) = \hat{U}(t)\hat{\rho}_{\rm S}(t)\hat{U}^{\dagger}(t), \tag{8.24}$$

$$\hat{U}(t) = \overrightarrow{T} \exp\left[\frac{\mathrm{i}}{\hbar}\int_0^t \left\{\hat{H}_{\rm S} + \hat{H}_{\rm R} + \hat{H}_{\rm ext}(\tau)\right\} \mathrm{d}\tau\right]. \tag{8.25}$$

Verify that $\hat{\rho}_{\rm SI}(t)$ obeys the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}_{SI}(t) = -\frac{\mathrm{i}}{\hbar}\left[\hat{H}_{\rm I}(t),\ \hat{\rho}_{SI}(t)\right], \tag{8.26}$$

$$\hat{H}_{\rm I}(t) = \hat{U}\hat{H}_{\rm RS}\hat{U}^{\dagger}. \tag{8.27}$$

The form of $\hat{H}_{\rm I}(t)$ is as in (8.17) with $\hat{F}_{kI}(t)$ as in (8.18) but $\hat{S}_{kI}(t)$ are now given by

$$\hat{S}_{kI}(t) = \overrightarrow{T} \exp\left[\frac{\mathrm{i}}{\hbar}\int_0^t \left\{\hat{H}_{\rm S} + \hat{H}_{\rm ext}(\tau)\right\} \mathrm{d}\tau\right] \hat{S}_k \\ \overleftarrow{T} \exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^t \left\{\hat{H}_{\rm S} + \hat{H}_{\rm ext}(\tau)\right\} \mathrm{d}\tau\right]. \tag{8.28}$$

The formal solution of (8.26) is the same as in (8.6). We can follow the steps leading from (8.6) to the master equation (8.20) by first rearranging $\hat{H}_{\rm I}(t)$ in the form:

$$\hat{H}_{\rm I}(t) = \hbar\sum_{k=1}^{N}\left(\exp\left(\mathrm{i}\tilde{\Omega}_k\ t\right)\hat{\tilde{S}}_k^{\dagger}\hat{\tilde{F}}_{kI} + \exp\left(-\mathrm{i}\tilde{\Omega}_k\ t\right)\hat{\tilde{F}}_{kI}^{\dagger}\hat{\tilde{S}}_k\right). \tag{8.29}$$

It then follows that the master equation in the present case would read

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}_{\rm S} = -\frac{\mathrm{i}}{\hbar}\left[\hat{H}_{\rm ext}(t),\ \hat{\rho}_{\rm S}\right] - \frac{\mathrm{i}}{\hbar}\left[\hat{H}_{\rm S}(t),\ \hat{\rho}_{\rm S}\right] + \hat{\tilde{L}}_{\rm S0}\hat{\rho}_{\rm S}, \tag{8.30}$$

with $\hat{\tilde{L}}_{\rm S0}$ given by (8.21) but with $\hat{S}_K \to \hat{\tilde{S}}_k$, $\hat{F}_{kI} \to \hat{\tilde{F}}_{kI}$, and $\Omega_k \to \tilde{\Omega}_k$. However, if $||\hat{H}_{\rm S}|| >> ||\hat{H}_{\rm ext}||$ then the time evolution in (8.28) may be assumed to be solely due to $\hat{H}_S$, i.e. $\hat{H}_{\rm ext}(t)$ in the exponentials in that expression may be ignored in comparison with $\hat{H}_{\rm S}$. This is the *weak external field approximation*. It reduces (8.28) to (8.19). The Liouvillean $\hat{\tilde{L}}_{S0}$ in (8.30) is then the same as $\hat{\tilde{L}}_{S0}$ in (8.21). In other words, the Liouvillean of the master equation of a system in a weak external field is the sum of the Liouvillean of its evolution in that field alone and that of its evolution in its absence.

Now, compare the exact formal expression for $\hat{\rho}_{\rm S}(t) = {\rm Tr}_{\rm R}[\hat{\rho}(t)]$, where $\hat{\rho}(t)$ is given by (8.2), with the formal solution of the master equation (8.20) which is in the Born–Markov approximation. Assuming, for the sake of simplicity, $\hat{\tilde{L}}_{\rm S}$ to be time-independent we infer that, in the Born–Markov approximation,

$$\mathbf{Tr_R}\left[\exp\left(\hat{\tilde{L}}t\right)\hat{\rho}(0)\right] \approx \exp\left(\hat{\tilde{L}}_{\rm S}t\right)\mathbf{Tr_R}\left[\hat{\rho}(0)\right]. \tag{8.31}$$

This relation will prove useful in finding multi-time averages in Sect. 8.3.

8.2 Solving a Master Equation

Consider a master equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho} = \hat{\hat{L}}\rho. \tag{8.32}$$

It may be solved by converting it in to a c-number equation by taking its matrix elements in an appropriately chosen basis. If $\hat{\hat{L}}$ is independent of time then the resulting equation may be solved by the method of eigenvectors exapansion outlined in the Chap. 10. In the problems of interest to us, one of the eigenvalues of $\hat{\hat{L}}$ is zero whereas the real part of all its other eigenvalues is negative. Then, in the limit $t \to \infty$, the system approaches the state $\hat{\rho}_{\mathrm{ss}}$, called its steady state. It solves

$$\hat{\hat{L}}\hat{\rho}_{\mathrm{ss}} = 0. \tag{8.33}$$

We assume that the solution of (8.33) is unique. The task of solving this equation is greatly facilitated if $\hat{\hat{L}}$ obeys the condition (5.41) of detailed balance. Along with the requirement $\tilde{\hat{\rho}}_{\mathrm{ss}} = \hat{\rho}_{\mathrm{ss}}$ (where $\tilde{\hat{A}}$ is time-reversed form of $\hat{A}$), that condition in the superoperator language may be rewritten as

$$\hat{\rho}_{\mathrm{ss}}\tilde{\hat{\hat{L}}}\hat{F} = \hat{\hat{L}}_{\mathrm{tr}}\hat{\rho}_{\mathrm{ss}}\hat{F}, \tag{8.34}$$

$\hat{F}$ being an arbitrary operator, $\hat{\hat{L}}_{\mathrm{tr}}$ is the time-reversed form of $\hat{\hat{L}}$ whereas $\tilde{\hat{\hat{L}}}$ is related with $\hat{\hat{L}}$ by

$$\mathrm{Tr}[\hat{A}\hat{\hat{L}}\hat{B}] = \mathrm{Tr}[\hat{B}\tilde{\hat{\hat{L}}}\hat{A}] \tag{8.35}$$

for all $\hat{A}$ and $\hat{B}$. See Chap. 12 for an application of the condition (8.34).

Alternatively, we may work with the equations for the expectation values of the operators. These equations are obtained by multiplying the master equation by the operator whose expectation value equation is desired, followed by performing the operation of trace. As an illustration, consider

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho} = -\mathrm{i}\left[\hat{B},\ \hat{\rho}\right] + 2\hat{A}\hat{\rho}\hat{A}^{\dagger} - \hat{A}^{\dagger}\hat{A}\hat{\rho} - \hat{\rho}\hat{A}^{\dagger}\hat{A}. \tag{8.36}$$

Notice that the evolution operator of any master equation is a sum of terms of the kind appearing on the right hand side of the equation above. The equation of motion for the expectation value $\langle\hat{O}\rangle = \mathrm{Tr}[\hat{O}\hat{\rho}]$ of an operator $\hat{O}$, obtained by multiplying (8.36) by $\hat{O}$ and on using the cyclic property of the trace reads

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{O}\rangle &= -\mathrm{i}\mathrm{Tr}\left\{\hat{O}\left[\hat{B},\ \hat{\rho}\right]\right\} + \mathrm{Tr}\left[\left\{2\hat{A}^{\dagger}\hat{O}\hat{A} - \hat{O}\hat{A}^{\dagger}\hat{A} - \hat{A}^{\dagger}\hat{A}\hat{O}\right\}\hat{\rho}\right] \\ &= -\mathrm{i}\mathrm{Tr}\left\{\hat{\rho}\left[\hat{O},\ \hat{B}\right]\right\} + \mathrm{Tr}\left[\left\{[\hat{A}^{\dagger},\hat{O}]\hat{A} - \hat{A}^{\dagger}[\hat{A},\hat{O}]\right\}\hat{\rho}\right] \\ &= -\mathrm{i}\left\langle\left[\hat{O},\ \hat{B}\right]\right\rangle + \left\langle[\hat{A}^{\dagger},\hat{O}]\hat{A} - \hat{A}^{\dagger}[\hat{A},\hat{O}]\right\rangle. \end{aligned} \tag{8.37}$$

The operator in the expectation value sign on the right hand side in the equation above may or may not be a linear combination of a constant and a constant multiple of $\hat{O}$. If it is not, then we need to derive equation for new operators arising therein. This process is carried till a closed set of equations is obtained.

The equation of motion of an operator may be put in a form suitable for applying to multi time averages. To that end, and for the sake of simplicity, let $\hat{\hat{L}}_S$ be time-independent. On using the formal solution of (8.20) and the definition (8.35), note that the expectation value of an operator $\hat{A}^{(S)}$ acting in the space of S alone may be written as

$$\begin{aligned}\langle \hat{A}^{(S)}(t)\rangle &= \mathrm{Tr}_S\left[\hat{A}^{(S)}\hat{\rho}_S(t)\right] = \mathrm{Tr}_S\left[\hat{A}^{(S)}\exp\left(\hat{\hat{L}}_S t\right)\hat{\rho}_S(0)\right] \\ &= \mathrm{Tr}_S\left[\hat{\rho}_S(0)\exp\left(\hat{\hat{L}}_S t\right)\hat{A}^{(S)}\right]. \end{aligned} \tag{8.38}$$

Note that $\hat{\hat{L}}$ acting on a system operator yields a linear combination of the system operators. We may, therefore, write

$$\hat{A}^{(S)}(t) \equiv \exp\left(\hat{\hat{L}}_S t\right)\hat{A}^{(S)} = \sum_m f_m(t)\hat{A}_m^{(S)}, \tag{8.39}$$

the $f_m(t)$ being c-number functions of time. On combining this with (8.38) we find that

$$\langle \hat{A}^{(S)}(t)\rangle = \sum_m f_m(t)\left\langle \hat{A}_m^{(S)}\right\rangle, \quad \left\langle \hat{A}_m^{(S)}\right\rangle = \mathrm{Tr}_S[\hat{\rho}_S(0)\hat{A}_m^{(S)}]. \tag{8.40}$$

Thus, the average of any system operator is determined by solving a closed set of equations for $\hat{A}_1, \hat{A}_2, \ldots$

Which of the approaches outlined above should be preferred depends on the situation at hand. For example, see Chap. 12.

However, solving a master equation analytically exactly is generally a formidable task. We outline the approximation methods for solving master equations for the situations encountered generally in quantum optics. Consider first a system in interaction with a reservoir and subject also to an external field. The system is then described by (8.30). If the interaction of the system with the external field is weak compared with its interaction with the reservoir, then we can carry a perturbation expansion of its solution in powers of $||\hat{H}_{\mathrm{ext}}||$. The details of this case are presented in the Chap. 9.

Consider next the case when, in an appropriate interaction picture, the hamiltonian part of the master equation (8.20) is independent of time. Transform the density operator $\hat{\rho}_S$ to

$$\hat{\rho}_I(t) = \exp\left\{\mathrm{i}\hat{H}_S t/\hbar\right\}\hat{\rho}_S \exp\left\{-\mathrm{i}\hat{H}_S t/\hbar\right\}. \tag{8.41}$$

The $\hat{\rho}_I$ obeys the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}_I = \hat{\hat{L}}_I(t)\hat{\rho}_I, \tag{8.42}$$

$$\hat{\hat{L}}_I(t) = \exp\left\{\mathrm{i}\hat{H}_{\mathrm{S}}t/\hbar\right\}\hat{\hat{L}}_{\mathrm{S0}}\exp\left\{-\mathrm{i}\hat{H}_{\mathrm{S}}t/\hbar\right\}. \tag{8.43}$$

The $\hat{\hat{L}}_I(t)$ will be of the form

$$\hat{\hat{L}}_I(t) = \hat{\hat{\Lambda}}_0 + \hat{\hat{\Lambda}}(t), \tag{8.44}$$

the $\hat{\hat{\Lambda}}_0$ being independent of time. The time-dependent part $\hat{\hat{\Lambda}}(t)$ would involve non-zero differences of the eigenvalues λ_i of $\hat{H}_{\mathrm{S}}$. The contribution to the solution of (8.42) from $\hat{\hat{\Lambda}}(t)$ part would, therefore, be of the order of $||\hat{\hat{L}}_{\mathrm{S0}}||/|\lambda_i - \lambda_j|$, $\lambda_i \neq \lambda_j$. Hence, if the differences in the eigenvalues is sufficiently large compared with $||\hat{\hat{L}}_{\mathrm{S0}}||$ then we can ignore $\hat{\hat{\Lambda}}(t)$ in comparison with $\hat{\hat{\Lambda}}_0$. The approximation wherein only the $\hat{\hat{\Lambda}}_0$ in (8.44) is retained is referred to as the *secular approximation*. As examples of its applications, see Sect. 12.2 and Sect. 14.2.

8.3 Multi-Time Average of System Operators

So far we have discussed the problem of solving a master equation for deriving operators averages at a single time t. The multi-time averages are also of experimental interest like in characterizing the spectrum and determining correlations between the dynamical variables at diferent times. Consider the two-time average of operators $A^{(S)}$ and $\hat{B}^{(S)}$ of S. By definition,

$$\left\langle \hat{A}^{(S)}(t+\tau)\hat{B}^{(S)}(t)\right\rangle = \mathrm{Tr}_{R+S}\left[\hat{A}^{(S)}(t+\tau)\hat{B}^{(S)}(t)\hat{\rho}(0)\right]. \tag{8.45}$$

The operators are in the Heisenberg picture generated by the combined hamiltonian $\hat{H}$. On invoking (1.77), and on applying the cyclic property of trace, (8.45) leads to

$$\begin{aligned}
&\left\langle \hat{A}^{(S)}(t+\tau)\hat{B}^{(S)}(t)\right\rangle \\
&= \mathrm{Tr}_{R+S}\left[\hat{A}^{(S)}(\tau)\hat{B}^{(S)}\exp\Big(-\frac{\mathrm{i}}{\hbar}\hat{H}t\Big)\hat{\rho}(0)\exp\Big(\frac{\mathrm{i}}{\hbar}\hat{H}t\Big)\right] \\
&= \mathrm{Tr}_{R+S}\left[\hat{A}^{(S)}\exp\Big(-\frac{\mathrm{i}}{\hbar}\hat{H}\tau\Big)\hat{B}^{(S)}\hat{\rho}(t)\exp\Big(\frac{\mathrm{i}}{\hbar}\hat{H}\tau\Big)\right] \\
&\equiv \mathrm{Tr}_{R+S}\left[\hat{A}^{(S)}\exp\Big(\hat{\hat{L}}\tau\Big)\Big(\hat{B}^{(S)}\hat{\rho}(t)\Big)\right].
\end{aligned} \tag{8.46}$$

The last line above is by virtue of the definition (8.2). Use (8.31) to reduces (8.46) in the Born–Markov approximation to

$$\left\langle \hat{A}^{(S)}(t+\tau)\hat{B}^{(S)}(t)\right\rangle = \mathrm{Tr}_S\left[\hat{A}^{(S)}\exp\left(\hat{\hat{L}}_S\tau\right)\left\{\hat{B}^{(S)}\hat{\rho}_S(t)\right\}\right]$$
$$= \mathrm{Tr}_S\left[\hat{B}^{(S)}\hat{\rho}_S(t)\exp\left(\hat{\hat{L}}_S\tau\right)\hat{A}^{(S)}\right]$$
$$= \sum_m f_m(\tau)\left\langle \hat{A}_m^{(S)}\hat{B}^{(S)}\right\rangle. \tag{8.47}$$

In writing the last line above, we have invoked (8.39). The two-time average is thus expressed in terms of the functions $f_m(t)$ determined by single time averages. This is the *quantum regression theorem.* It is the quantum analog of the regression theorem (5.38) for a classical Markov process. Similar results may be derived also for multi-time averages [68].

Next, we specialise the master equation by fixing the choice of a reservoir.

8.4 Bath of Harmonic Oscillators

A reservoir of common interest is the one constituted by an infinite number of harmonic oscillators. An example of such a reservoir is the e.m. field in free space. Its interaction with an atom leads to the phenomenon of spontaneous emission from an excited atomic level. The infinite system of harmonic oscillators may even be used to model the motion of atoms in the walls of a cavity. The system S then is a mode of the cavity field. The reservoir in this case causes decay of the cavity modes at the walls of the cavity.

Let $\hat{b}_k^\dagger$, $\hat{b}_k$ be the creation and annihilation operators of a harmonic oscillators of frequency ω_k constituting the reservoir. The free reservoir hamiltonian is then given by

$$\hat{H}_R = \hbar\sum_k \omega_k \hat{b}_k^\dagger \hat{b}_k. \tag{8.48}$$

The hamiltonian of the combined system of the reservoir and the system S interacting with it then reads

$$\hat{H} = \hat{H}_S + \hbar\sum_k \omega_k \hat{b}_k^\dagger \hat{b}_k + \hbar\sum_i \left[\hat{F}_i \hat{S}_i^\dagger + \hat{F}_i^\dagger \hat{S}_i\right]. \tag{8.49}$$

We assume that the reservoir operator $\hat{F}_i$ appearing in the interaction hamiltonian above is a linear function of $\hat{b}_k$ and $\hat{b}_k^\dagger$. Its form depends on whether or not we choose to work in the rotating wave approximation (RWA) introduced in Chap. 7. Assuming $\hat{S}_i$ to be a lowering operator, the form of $\hat{F}_i$ in the RWA is

$$\hat{F}_i = \sum_k g_{ik}\hat{b}_k, \tag{8.50}$$

the g_{jk} being the system-reservoir coupling constant. Without the RWA,

$$\hat{F}_i = \sum_k \left[g_{ik}\hat{b}_k + g_{ik}^*\hat{b}_k^\dagger \right]. \tag{8.51}$$

The $\hat{F}_{kI}$ defined in (8.18) are found by noting that

$$\hat{b}_{kI}(t) = \exp\left(\mathrm{i}\hat{H}_\mathrm{R}\ t/\hbar\right)\hat{b}_k \exp\left(-\mathrm{i}\hat{H}_\mathrm{R}\ t/\hbar\right) = \exp(-\mathrm{i}\omega_k t)\hat{b}_k. \tag{8.52}$$

Now, in order to evaluate the rates defined in (8.22), we need to fix the state of the reservoir. The reservoirs of common interest are: (i) a *thermal reservoir*, and (ii) a *squeezed reservoir*.

8.4.1 Thermal Reservoir

Thermal reservoir is constituted by harmonic oscillators in the state of equilibrium at themperature T. It is described by the density operator

$$\hat{\rho}_\mathrm{th} = \prod_{k=1}(1 - \exp(-\beta_k)) \exp\left[-\beta_k \hat{b}_k^\dagger \hat{b}_k\right], \tag{8.53}$$

$\beta_k = \hbar\omega_k/k_\mathrm{B}T$, k_B being the Boltzmann constant. The correlation functions needed for evaluating the rates (8.22) are easily found by applying the formulae (6.79)-(6.81). Verify that ($\hat{b}(\omega_k) \equiv \hat{b}_k$)

$$\begin{aligned}
\langle \hat{b}(\omega_k)\rangle &= \langle \hat{b}^\dagger(\omega_k)\rangle = 0, \\
\langle \hat{b}^\dagger(\omega_k)\hat{b}(\omega_l)\rangle &= \bar{n}(\omega_k)\delta\left(\omega_k - \omega_l\right), \quad \bar{n}(\omega_k) = \frac{1}{\exp(\beta_k) - 1}, \\
\langle \hat{b}(\omega_l)\hat{b}^\dagger(\omega_k)\rangle &= (\bar{n}(\omega_k) + 1)\,\delta\left(\omega_k - \omega_l\right), \\
\langle \hat{b}(\omega_k)\hat{b}(\omega_l)\rangle &= \langle \hat{b}^\dagger(\omega_k)\hat{b}^\dagger(\omega_l)\rangle = 0.
\end{aligned} \tag{8.54}$$

Consider first the case of interaction in the RWA. On using (8.18) read with (8.50) and (8.52) we find that

$$\begin{aligned}
\langle \hat{F}_{jI}^\dagger(t-\tau)\hat{F}_{iI}(t)\rangle &= \sum_k g_{jk}^* g_{ik} \exp(-\mathrm{i}\omega_k\tau)\bar{n}(\omega_k), \\
\langle \hat{F}_{jI}(t-\tau)\hat{F}_{iI}^\dagger(t)\rangle &= \sum_k g_{jk}^* g_{ik} \exp(\mathrm{i}\omega_k\tau)(\bar{n}(\omega_k) + 1), \\
\langle \hat{F}_{jI}(t-\tau)\hat{F}_{iI}(t)\rangle &= \langle \hat{F}_{jI}^\dagger(t)\hat{F}_{iI}^\dagger(t-\tau)\rangle = 0.
\end{aligned} \tag{8.55}$$

Now, let the frequencies be continuously distributed and let $h(\omega)$ be the density of oscillators at frequency ω. We can then convert a sum over k to an integral over ω by means of the correspondence

$$\sum_k f_k \to \int_0^\infty \mathrm{d}\omega h(\omega) f(\omega). \tag{8.56}$$

Derivation of $W^{(1,2)}$ involves evaluation of terms of the kind,

$$I_{\pm} = \int_0^{\infty} \mathrm{d}\tau \sum_k \exp\left(\pm \mathrm{i}\left(\omega_k - \Omega\right)\tau\right) f_k$$
$$\to \int_0^{\infty} \mathrm{d}\tau \int_0^{\infty} \mathrm{d}\omega h(\omega) \exp\left\{\pm \mathrm{i}\left(\omega - \Omega\right)\tau\right\} f(\omega). \tag{8.57}$$

On recalling (A.3), this reduces to

$$I_{\pm} = \int_0^{\infty} \mathrm{d}\omega\ h(\omega)\ f(\omega) \left\{\pi\delta\left(\omega - \Omega\right) \pm \mathrm{i}P\left(\frac{1}{\omega - \Omega}\right)\right\}$$
$$= \pi h(\Omega) f\left(\Omega\right) \pm \mathrm{i}P \int d\omega \frac{h(\omega) f(\omega)}{\omega - \Omega}. \tag{8.58}$$

On applying these results, (8.22) yields $W_{ij}^{(3)} = W_{ij}^{(4)} = 0$ and

$$W_{ij}^{(1)} = \gamma_{ij}\left(\Omega_i\right)\bar{n}(\Omega_i) + \mathrm{i}\tilde{\Omega}_{ij}^{(1)},$$
$$W_{ij}^{(2)} = \gamma_{ij}\left(\Omega_i\right)\left(\bar{n}(\Omega_i) + 1\right) - \mathrm{i}\tilde{\Omega}_{ij}^{(2)}, \tag{8.59}$$

$$\gamma_{ij}\left(\Omega_i\right) = \pi \sum_k g_{ik}^* g_{jk} \delta(\omega_k - \Omega_i),$$
$$\tilde{\Omega}_{ij}^{(1)} = P \sum_k \frac{g_{ik}^* g_{jk} \bar{n}(\omega_k)}{\Omega_i - \omega_k}. \tag{8.60}$$

The $\tilde{\Omega}_{ij}^{(2)}$ is obtained by replacing $\bar{n}(\omega_k)$ in $\tilde{\Omega}_{ij}^{(1)}$ by $\bar{n}(\omega_k) + 1$. Substitution of these rates in (8.21) gives the master equation for a system interacting with a reservoir of harmonic oscillators in thermal equilibrium at temperature T in RWA.

Without the RWA, the reservoir operator is given by (8.51). The rates $W_{ij}^{(3,4)}$ are now non-zero. Note that in the interaction picture defined in (8.7), the terms multiplying these rates in the equation for $\hat{\rho}_{SI}(t)$ oscillate at $\Omega_i + \Omega_j$. The time t of observation is usually very large compared with $1/\Omega_k$. Hence, contribution of these terms averages to zero on the time scale of observation. The terms multiplying the rates $W_{ij}^{(3,4)}$ may, therefore, be ignored. This amounts to making the RWA on the master equation. The resulting master equation, like the one derived by making the RWA on the hamiltonian, does not contain terms corresponding to $W_{lk}^{(3,4)}$. However, verify that $W_{lk}^{(1,2)}$ now involve in addition to $I_{\pm}$ defined in (8.57), the terms of the kind

$$J_{\pm} = \int_0^{\infty} \mathrm{d}\tau \sum_k \exp\left(\pm \mathrm{i}\left(\omega_k + \Omega\right)\tau\right) f_k$$
$$\to \int_0^{\infty} d\omega\ h(\omega)\ f(\omega) \left\{\pi\delta\left(\omega + \Omega\right) \pm \mathrm{i}P\left(\frac{1}{\omega + \Omega}\right)\right\}$$
$$= \pm \mathrm{i}P \int d\omega \frac{h(\omega) f(\omega)}{\omega + \Omega}. \tag{8.61}$$

These terms contribute to the principal part in $W_{ij}^{(1,2)}$. We will see in the examples below that the principal part contributes to reversible part of the evolution. Thus, the master equation obtained by employing the hamiltonian without the RWA but making it on the master equation is the same as the one obtained by using the hamiltonian in the RWA except in the modification of frequencies of the rversible evolution.

8.4.2 Squeezed Reservoir

A squeezed reservoir is characterized by the density operator

$$\hat{\rho}_{\rm sq} = \hat{S}(\xi)\hat{\rho}_{\rm th}\hat{S}^{\dagger}(\xi) \tag{8.62}$$

where $\hat{\rho}_{\rm th}$ is given by (8.53), and

$$\hat{S}^{\dagger}(\xi) = \prod_k \exp\left(\xi^* \hat{b}(\Omega_p - \omega_k)\hat{b}(\Omega_p + \omega_k) - \text{h.c.}\right) \tag{8.63}$$

is the so called *two-mode squeezing operator*. It is an element of $SU(1,1)$ group. The correlation functions needed for evaluating the rates may be derived by noting that, for any $\hat{A}$, $\mathrm{Tr}[\hat{\rho}_{\rm sq}\hat{A}] = \mathrm{Tr}[\hat{\rho}_{\rm th}\hat{S}^{\dagger}(\xi)\hat{A}\hat{S}(\xi)]$. Use the results of Chap. 2 to show that

$$\begin{aligned}
&\hat{S}^{\dagger}(\xi)\hat{b}(\Omega_p \pm \omega_k)\hat{S}(\xi) \\
&= \cosh(|\xi|)\hat{b}(\Omega_p \pm \omega_k) + \exp(\mathrm{i}\phi)\sinh(|\xi|)\hat{b}^{\dagger}(\Omega_p \mp \omega_k),
\end{aligned} \tag{8.64}$$

$\xi = |\xi|\exp(\mathrm{i}\phi)$. It is now straightforward to verify that

$$\begin{aligned}
\langle \hat{b}(\omega_k)\rangle &= \langle \hat{b}^{\dagger}(\omega_k)\rangle = 0, \\
\langle \hat{b}^{\dagger}(\omega_k)\hat{b}(\omega_l)\rangle &= \bar{N}(\omega_k)\delta\left(\omega_k - \omega_l\right), \\
\langle \hat{b}(\omega_l)\hat{b}^{\dagger}(\omega_k)\rangle &= \left(\bar{N}(\omega_k) + 1\right)\delta\left(\omega_k - \omega_l\right), \\
\langle \hat{b}(\omega_k)\hat{b}(\omega_l)\rangle &= \bar{M}(\omega_k)\delta\left(\omega_k + \omega_l - 2\Omega_p\right),
\end{aligned} \tag{8.65}$$

$$\begin{aligned}
\bar{N} &= \bar{n}(\cosh^2(|\xi|) + \sinh^2(|\xi|)) + \sinh^2(|\xi|), \\
\bar{M} &= (2\bar{n} + 1)\sinh(|\xi|)\cosh(|\xi|)\exp(\mathrm{i}\phi).
\end{aligned} \tag{8.66}$$

Verify that

$$\bar{N}(\bar{N} + 1) = |\bar{M}|^2 + \bar{n}(\bar{n} + 1). \tag{8.67}$$

For $\xi = 0$ the relations (8.66) reduce to those of a thermal bath. On the other hand if $\bar{n} = 0$ then $\rho_{\rm th} = |\{0\}\rangle\langle\{0\}|$, i.e. $\rho_{\rm th}$ then describes the vacuum state. The squeezed bath is consequently in a pure state, $\hat{S}(\xi)|\{0\}\rangle$, called the *squeezed vacuum* .

We evaluate the rates by treating the interaction in the RWA. The rates $W_{ij}^{(1,2)}$ are given by (8.59) with $\bar{n} \to \bar{N}$:

$$W_{ij}^{(1)} = \gamma_{ij}\bar{N}(\Omega_i) + \mathrm{i}\tilde{\Omega}_{ij}^{(1)}, \quad W_{ij}^{(2)} = \gamma_{ij}\left(\bar{N}(\Omega_i) + 1\right) - \mathrm{i}\tilde{\Omega}_{ij}^{(2)}. \tag{8.68}$$

Ignoring the principal part,

$$W_{ij}^{(3)} = W_{ij}^{(4)*} = \tilde{\gamma}_{ij}(\Omega_i)\bar{M}(\Omega_i)\exp(-2\mathrm{i}\Omega_p t), \tag{8.69}$$

$$\tilde{\gamma}_{ij}(\Omega_i) = \pi \sum_k g_i^*(\omega_k) g_j(2\Omega_p - \omega_k)\delta(\omega_k - \Omega_i). \tag{8.70}$$

The master equation of a system interacting with a squeezed bath of harmonic oscillators is determined by substituting the rates derived above in (8.21).

8.4.3 Reservoir of the Electromagnetic Field

As stated before, a realization of a harmonic oscillator bath of particular interest to us is the one constituted by the e.m. field in free space. The properties of the e.m. field are of paramount interest. For the sake of illustration, we consider an oscillator of frequency ω_0 at $\boldsymbol{r}_0$ interacting with the e.m. field in free space. Corresponding interaction hamiltonian in the RWA, by virtue of (7.44), may be written as

$$\hat{H}_{\mathrm{RS}} = \hbar \sum_k \left[g_k^* b_k^\dagger \hat{S} + g_k \hat{S}^\dagger \hat{b}_k \right], \tag{8.71}$$

$$g_k = -\mathrm{i}\sqrt{\frac{\omega_k}{2\hbar V}}\, \boldsymbol{d}.\boldsymbol{\epsilon}_\lambda \exp(\mathrm{i}\boldsymbol{k}.\boldsymbol{r_0}), \tag{8.72}$$

the $\boldsymbol{d}$ being the electric dipole moment of the oscillator. The free evolution of the oscillator is described by

$$\hat{S}(t) = \exp(-\mathrm{i}\omega_0 t)\hat{S}. \tag{8.73}$$

The hamiltonian (8.71) is of the form (8.49) read with (8.50). Now, the solution of the Heisenberg equation for $\hat{b}_k$ is

$$\hat{b}_k(t) = \exp(-\mathrm{i}\omega_k\, t)\, \hat{b}_k(0) - \mathrm{i}\int_0^t \mathrm{d}\tau g_k^* \exp(-\mathrm{i}\omega_k\, (t-\tau))\, \hat{S}(\tau). \tag{8.74}$$

The field at any position $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}_0$ may be found by inserting (8.74) in (6.25). Recall that the summation over k in that expression stands for summation over the wave vector and the polarization directions:

$$\frac{1}{V}\sum_k \equiv \frac{1}{V}\sum_{\boldsymbol{k}}\sum_\lambda \to \frac{1}{(2\pi)^3}\int \mathrm{d}^3k \sum_\lambda. \tag{8.75}$$

The last expression above is the integral equivalent of the summation over $\boldsymbol{k}$ in the continuum limit. The summation over λ and integration over $\boldsymbol{k}$ yields, to order $|\boldsymbol{R}|^{-1}$,

$$\boldsymbol{E}^{(+)}(\boldsymbol{R},t) = \boldsymbol{E}_0^{(+)}(\boldsymbol{R},t) - k_0^2\Big[\boldsymbol{R}\times(\boldsymbol{R}\times\boldsymbol{d})|\boldsymbol{R}|^{-3} \times \exp\left(-\mathrm{i}k_0\boldsymbol{r}.\boldsymbol{r}_0/|\boldsymbol{r}|\right)\hat{S}(t-|\boldsymbol{r}|/c)\Big]. \tag{8.76}$$

Here $k_0 = \omega_0/c$. We refer the reader to [93] for the details of the derivation. See also [89, 94].

The expression (8.76) for the field in the *radiation zone* determines the field in terms of the oscillator dynamics. The dynamical prperties of the oscillator are determined, of course, by solving the corresponding master equation. As an application of (8.76), we note that the two-time correlation function of the field in the far-field zone is

$$\langle \boldsymbol{E}^{(-)}(\boldsymbol{R},t).\boldsymbol{E}^{(+)}(\boldsymbol{R},t+\tau)\rangle \sim \langle \hat{S}^\dagger(t-|\boldsymbol{r}|/c)\hat{S}(t-|\boldsymbol{r}|/c+\tau)\rangle. \tag{8.77}$$

This, on substitution in (6.90) shows that the spectrum of the radiation is determined by

$$S(\omega) \sim \mathrm{Re}\int_0^\infty \mathrm{d}\tau \exp(\mathrm{i}\omega\tau)\langle \hat{S}^\dagger \hat{S}(\tau)\rangle. \tag{8.78}$$

In following two sections we derive the master equation for some atomic systems in contact with the reservoir of the e.m. field.

8.5 Master Equation for a Harmonic Oscillator

The system treated in this section is that of a harmonic oscillator of frequency ω_0 described by the hamiltonian

$$\hat{H}_\mathrm{S} = \hbar\omega_0\hat{a}^\dagger\hat{a}. \tag{8.79}$$

Let its interaction with the reservoir be described by

$$\hat{H}_\mathrm{RS} = \hbar\sum_k \left(g_k^*\hat{b}_k^\dagger\hat{a} + g_k\hat{a}^\dagger\hat{b}_k\right). \tag{8.80}$$

The hamiltonian of the combined system $S+R$ is then given by (8.49) with $\hat{S}_i \to \hat{a}$ and $\hat{F}_i$ given by (8.50). The correspondibg rates are given by (8.68) and (8.69). We assume that $\Omega_p \approx \omega_0$ and invoke the fact that g_k varies little around ω_0. We, therefore, have

$$\tilde{\gamma}(\omega_0) = \gamma(\omega_0). \tag{8.81}$$

The master equation (8.20) then assumes the form

$$\begin{aligned}\dot{\hat{\rho}}(t) = &-\mathrm{i}\omega\left[\hat{a}^\dagger\hat{a},\ \hat{\rho}\right] + \gamma\left(\bar{N}+1\right)\left[2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\right]\\ &+\gamma\bar{N}\left[2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{a}^\dagger - \hat{a}\hat{a}^\dagger\hat{\rho}\right]\\ &+\gamma\Big\{\bar{M}\exp(2\mathrm{i}\Omega_p t)\left[2\hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^2 - \hat{a}^2\hat{\rho}\right] + \mathrm{h.c.}\Big\}.\end{aligned} \tag{8.82}$$

Here ω is renormalized to include contribution from the principal parts of $W_{lk}^{(1,2)}$ and $\bar{N} \equiv \bar{N}(\omega_0), \bar{M} \equiv \bar{M}(\omega_0)$.

The equation (8.82) is obtained by eliminating the bath variables in the Born–Markov approximation. However, the bath variables in this model can be eliminated exactly. In order to see it, note that the Heisenberg equations for the system and the bath operators reading

$$\mathrm{i}\dot{\hat{a}} = \omega_0 \hat{a} + \sum_k g_k \hat{b}_k, \qquad \mathrm{i}\dot{\hat{b}}_k = \omega \hat{b}_k + g_k^* \hat{a} \tag{8.83}$$

are linear. These may be solved by the method of Laplace transformation. Let $\tilde{F}(z)$ denote the Laplace transform of $F(t)$ defined by

$$\tilde{F}(z) = \int_0^\infty \exp(-zt) F(t) \mathrm{d}t. \tag{8.84}$$

Then

$$F(t) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma - \mathrm{i}\infty}^{\gamma + \mathrm{i}\infty} \exp(zt) \tilde{F}(z) \mathrm{d}z, \tag{8.85}$$

where γ is such that the singularities of $f(z)$ lie on the left of the line at a distance γ parallel to the imaginary axis. Use the property

$$\int_0^\infty \exp(-zt) \dot{F}(t) \mathrm{d}t = -F(0) + z\tilde{F}(z) \tag{8.86}$$

to transform (8.83) to

$$(z + \mathrm{i}\omega_0)\tilde{\hat{a}} = \hat{a}(0) - \mathrm{i} \sum_k g_k \tilde{\hat{b}}_k, \qquad (z + \mathrm{i}\omega_k)\tilde{\hat{b}}_k = \omega_k \hat{b}_k(0) - \mathrm{i} g_k^* \hat{a}. \tag{8.87}$$

Eliminate $\tilde{\hat{b}}_k$ from these equations to express $\tilde{\hat{a}}$ in terms of $\hat{a}(0)$ and $\{\hat{b}_k(0)\}$. The inverse transform of the resulting expression for $\tilde{\hat{a}}$ gives

$$\hat{a}(t) = f(t)\hat{a}(0) + \sum_k \phi_k(t) \hat{b}_k(0) \tag{8.88}$$

where

$$\begin{aligned} f(t) &= \frac{1}{2\pi \mathrm{i}} \int_{\gamma - \mathrm{i}\infty}^{\gamma + \mathrm{i}\infty} \mathrm{d}z \frac{\exp(zt)}{\eta(z)}, \\ \phi_k(t) &= -\frac{g_k}{2\pi} \int_{\gamma - \mathrm{i}\infty}^{\gamma + \mathrm{i}\infty} \mathrm{d}z \frac{\exp(zt)}{(z + \mathrm{i}\omega_k)\eta(z)}, \end{aligned} \tag{8.89}$$

$$\eta(z) = z + \mathrm{i}\omega_0 + \sum_{k=1}^{N} \frac{|g_k|^2}{z + \mathrm{i}\omega_k}. \tag{8.90}$$

Clearly, any function of $\hat{a}(t); \hat{a}^\dagger(t)$ may be evaluated using (8.88). For an exact evaluation of the density operator see [92]. The nature of evolution is determined by the nature of the time-dependence of $f(t)$ and $\phi_k(t)$ which, in turn, is dictated by the roots of $\eta(z)$. If the number N of oscillators in the reservoir is finite then $\eta(z) = 0$ reduces to a polynomial of degree $N +$

1 in z. Its roots are isolated poles on the imaginary axis. The functions $f(t)$ and $\phi_k(t)$ are then sums of oscillatory functions of time. As a result, S executes a reversible motion. However, in the limit of continuous distribution of frequencies, it may be shown that

$$f(t) \sim \exp(-\mathrm{i}\omega t - \gamma t). \tag{8.91}$$

The $\phi_k(t)$, obtained by substituting (8.91) in (8.89) also acquires a damping part. The motion is thus irrevrsible in the limit $N \to \infty$. In this limit, the density operator $\hat{\rho}_{\mathrm{S}}(t)$ evolves according to (8.82). For details, refer to [92].

8.6 Master Equation for Two-Level Atoms

Consider a system of N identical two-level atoms each of frequency ω_0 interacting with the radiation field. The evolution of the combined system is governed by the hamiltonian

$$\hat{H} = \hbar\omega_0 \sum_{i=1}^{N} \hat{S}_z^{(i)} + \hbar \sum_k \omega_k \hat{a}_k^\dagger \hat{a}_k + \hbar \sum_{i=1}^{N} \left[\hat{S}_+^{(i)} \hat{F}_i + \hat{F}_i^\dagger \hat{S}_-^{(i)} \right]. \tag{8.92}$$

Here $\hat{S}_\pm^{(i)}$ are the raising and lowering operators for the i^{th} atom and $\hat{F}_i$ is given by (8.50). By identifying $\hat{S}_i$ in (8.16) as $\hat{S}_-^{(i)}$, it turns out that Ω_k in (8.19) are ω_0. Invoking also the (8.68) and (8.69), the master equation (8.20) assumes the form

$$\begin{aligned}
\dot{\hat{\rho}} = -\mathrm{i}\Big[& \omega_0 \sum_{i=1}^{N} \hat{S}_z^{(i)} + \sum_{i,j=1}^{N} \Omega_{ij} \hat{S}_+^{(i)} \hat{S}_-^{(j)},\ \hat{\rho} \Big] \\
& + \sum_{i,j=1}^{N} \gamma_{ij} \Big[(\bar{N}+1) \Big(2\hat{S}_-^{(j)} \hat{\rho} \hat{S}_+^{(i)} - \hat{\rho} \hat{S}_+^{(i)} \hat{S}_-^{(j)} - \hat{S}_+^{(i)} \hat{S}_-^{(j)} \hat{\rho} \Big) \\
& + \bar{N} \Big(2\hat{S}_+^{(j)} \hat{\rho} \hat{S}_-^{(i)} - \hat{\rho} \hat{S}_-^{(i)} \hat{S}_+^{(j)} - \hat{S}_-^{(i)} \hat{S}_+^{(j)} \hat{\rho} \Big) \\
& + \Big\{ \bar{M} \exp(2\mathrm{i}\Omega_p t) \Big(2\hat{S}_-^{(j)} \hat{\rho} \hat{S}_-^{(i)} - \hat{\rho} \hat{S}_-^{(i)} \hat{S}_-^{(j)} - \hat{S}_-^{(i)} \hat{S}_-^{(j)} \hat{\rho} \Big) + \mathrm{h.c.} \Big\} \Big].
\end{aligned} \tag{8.93}$$

The rates γ_{ij}, $\Omega_{ij} = \Omega_{ij}^{(1)} - \Omega_{ij}^{(2)}$ are as in (8.60). The g_{ij} in those equations are as in (8.72) (with $\boldsymbol{r}_0$ there replaced by the position $\boldsymbol{r}_i$ of the i^{th} atom and $g_k \to g_{ik}$). For the details of evaluation of the rates see, for example, [89]. For a single two-level atom,

$$\gamma = \frac{2}{3} |d_{12}|^2 \frac{\omega_0^3}{c^3}, \tag{8.94}$$

the $|d_{12}|$ being the dipole moment between the atomic levels in question.

The indices i, j in γ_{ij} and Ω_{ij} indicate their dependence on the atomic position through the factor $\exp(\boldsymbol{k}.(\boldsymbol{r}_i - \boldsymbol{r}_j))$. Due to the presence of the

delta function in the expression for γ_{ij}, the magnitude of $\boldsymbol{k}$ is restricted to $|\boldsymbol{k}_0| = \omega_0/c$. Hence, if $\boldsymbol{k}.(\boldsymbol{r}_i - \boldsymbol{r}_j) \ll 1$ for all i and j, i.e. if the atomic sample is confined to a region of dimensions small compared with the wavelength corresponding to the atomic transition frequency, then γ_{ij} become independent of space. The master equation (8.93) in the small atomic sample size approximation reduces to

$$\begin{aligned}\dot{\hat{\rho}} = &-\mathrm{i}\Big[\omega_0 \hat{S}_z + \sum_{i,j=1}^{N} \Omega_{ij} \hat{S}_+^{(i)} \hat{S}_-^{(j)},\ \hat{\rho}\Big] \\ &+\gamma\Big[(\bar{N}+1)\Big(2\hat{S}_-\hat{\rho}\hat{S}_+ - \hat{\rho}\hat{S}_+\hat{S}_- - \hat{S}_+\hat{S}_-\hat{\rho}\Big) \\ &+\bar{N}\Big(2\hat{S}_+\hat{\rho}\hat{S}_- - \hat{\rho}\hat{S}_-\hat{S}_+ - \hat{S}_-\hat{S}_+\hat{\rho}\Big) \\ &+\Big\{\bar{M}\exp(2\mathrm{i}\Omega_p t)\Big(2\hat{S}_-\hat{\rho}\hat{S}_- - \hat{\rho}\hat{S}_-\hat{S}_- - \hat{S}_-\hat{S}_-\hat{\rho}\Big) + \mathrm{h.c.}\Big\}\Big]. \qquad (8.95)\end{aligned}$$

Here $\hat{S}_\mu$ are collective atomic operators defined in (3.73).

We have thus at hand the equation governing the evolution of a system of two-level atoms interacting with the reservoir of the e.m. field. Now, let the atoms interact also with an external field whose action is described by $\hat{H}_{\mathrm{ext}}$. As discussed in Sect. 8.1, if the field is weak then the master equations derived above acquires an additional term reading $-\mathrm{i}[\hat{H}_{\mathrm{ext}}/\hbar, \hat{\rho}]$. However, if the field is strong then the master equation is derived by following the procedure outlined in Sect. 8.1. We illustrate that procedure by treating the example of a single two-level atom in a monochromatic external field.

8.6.1 Two-Level Atom in a Monochromatic Field

We consider a two-level atom in contact with the reservoir of the e.m. field and driven also by a monochromatic field of frequency ω. This interaction descries the phenomenon of *resonance fluorescence*. Let the interaction with the applied field be described by

$$\hat{H}_{\mathrm{ext}} = \hbar[\alpha \exp(-\mathrm{i}\omega t)\hat{S}_+ + \alpha^* \exp(\mathrm{i}\omega t)\hat{S}_-]. \qquad (8.96)$$

We derive the atomic master equation assuming, for the sake of simplicity, $\omega = \omega_0$. To that end, we need to evaluate (8.27):

$$\begin{aligned}\hat{H}_{\mathrm{I}}(t) &= \overrightarrow{T} \exp\left[\frac{\mathrm{i}}{\hbar}\int_0^t \mathrm{d}\tau \left\{\hat{H}_{\mathrm{S}} + \hat{H}_{\mathrm{ext}}(\tau)\right\}\right]\left[\hat{F}_I^\dagger(t)\hat{S}_- + \hat{F}_I(t)\hat{S}_+\right] \\ &\quad \times \overleftarrow{T} \exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^t \mathrm{d}\tau \left\{\hat{H}_{\mathrm{S}} + \hat{H}_{\mathrm{ext}}(\tau)\right\}\right] \\ &= \exp\left(\frac{\mathrm{i}}{\hbar}\hat{\tilde{H}}_{\mathrm{ext}} t\right)\exp\left(\frac{\mathrm{i}}{\hbar}\hat{H}_{\mathrm{S}} t\right)\left[\hat{F}_I^\dagger(t)\hat{S}_- + \hat{F}_I(t)\hat{S}_+\right] \\ &\quad \times \exp\left(-\frac{\mathrm{i}}{\hbar}\hat{H}_{\mathrm{S}} t\right)\exp\left(-\frac{\mathrm{i}}{\hbar}\hat{\tilde{H}}_{\mathrm{ext}} t\right), \qquad (8.97)\end{aligned}$$

$$\hat{\tilde{H}}_{\rm ext} = \hbar[\alpha \hat{S}_+ + \alpha^* \hat{S}_-]. \tag{8.98}$$

In writing the second line in (8.97) we have applied (1.51). For simplicity, we let α to be real. The similarity transformation in (8.97) is a special case of (7.29) corresponding to $\delta = 0$. Using that expression (or by evaluating (8.97) directly by applying (2.42)) we find that

$$\hat{H}_{\rm I}(t) = \frac{1}{2}\Big[\hat{F}_+(t)\hat{S}_+ + \hat{F}_-(t)\cos(2\alpha t)\hat{S}_- + {\rm i}\hat{F}_-(t)\sin(2\alpha t)\hat{S}_z\Big] + {\rm h.c.}, \tag{8.99}$$

$$\hat{F}_\pm(t) = \hat{F}_I^\dagger(t)\exp(-{\rm i}\omega_0 t) \pm {\rm h.c.}. \tag{8.100}$$

The $\hat{H}_{\rm I}(t)$ is of the form (8.17). Assuming that the reservoir is in the state of vacuum, the Liouvillean (8.21), ignoring the frequency shifts, assumes the form

$$\begin{aligned}\hat{\hat{L}}_{S0}\hat{\rho} = &-\frac{\rm i}{\hbar}\left[\hat{H}_{\rm ext}(t), \hat{\rho}\right] + \gamma_+\left(2\hat{S}_-\hat{\rho}\hat{S}_+ - \hat{S}_+\hat{S}_-\hat{\rho} - \hat{\rho}\hat{S}_+\hat{S}_-\right) \\ &+\Big\{\gamma_-\left(\hat{S}_+\hat{\rho}\hat{S}_+ - \hat{S}_+\hat{S}_+\hat{\rho}\right) \\ &+\tilde{\gamma}\left(\hat{S}_z\hat{\rho}\hat{S}_+ - \hat{S}_+\hat{S}_z\hat{\rho}\right) + {\rm h.c.}\Big\},\end{aligned} \tag{8.101}$$

$$\begin{aligned}\gamma_\pm &= \frac{1}{4}\left[2\gamma(\omega_0) \pm \{\gamma(\omega_0 + 2\alpha) + \gamma(\omega_0 - 2\alpha)\}\right], \\ \tilde{\gamma} &= \frac{1}{2}\left\{\gamma(\omega_0 + 2\alpha) - \gamma(\omega_0 - 2\alpha)\right\}.\end{aligned} \tag{8.102}$$

Note that if $|\alpha| \ll \omega_0$ (i.e. if $||\hat{H}_{\rm S}|| \gg ||\hat{H}_{\rm ext}||$) then the non-commutator part in (8.101) reduces to the vacuum field version ($\bar{N} = \bar{M} = 0$) of (8.95) without the frequency shift. It is in accordance with the assertion that the Liouvillean of the master equation of a system in a weak external field is a sum of the Liouvillean of its evolution in that field alone and that of its evolution in the absence of the field.

8.6.2 Collisional Damping

The master equation (8.95) describes radiative decay of an excited atomic level. It is caused by the resrvoir of the e.m. field. There are, however, other mechanisms responsible for the decay of an atomic level. One such mechanism of considerable interest is collisions of an atom with other atoms. Let us consider the collisions that alter the phase of the atomic state but not its population. Such collisions may be incorporated by assuming that the evolution of the atom is governed by the hamiltonian

$$\hat{H} = \hbar\left(\omega_0 + \mu(t)\right)\hat{S}_z. \tag{8.103}$$

The equation of motion of the atomic density operator then reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}(t) = -\mathrm{i}\omega_0\left[\hat{S}_z,\ \hat{\rho}(t)\right] + \hat{\hat{L}}_0\hat{\rho}(t) - \mathrm{i}\mu(t)\left[\hat{S}_z,\ \hat{\rho}(t)\right]. \tag{8.104}$$

Here $\hat{\hat{L}}_0$ accounts for any other interaction that the atom may be involved in. On assuming that $\mu(t)$ is a delta correlated Gaussian process with zero mean, it follows by applying the methods of Chap. 5 that the density operator averaged over collisional fluctuations obeys

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\bar{\rho}}(t) = -\mathrm{i}\omega\left[\hat{S}_z,\ \hat{\rho}(t)\right] + \hat{\hat{L}}_0\hat{\bar{\rho}}(t) + \hat{\hat{L}}_c\hat{\bar{\rho}}(t) \tag{8.105}$$

where

$$\hat{\hat{L}}_c\hat{\bar{\rho}}(t) = -\gamma_c[\hat{S}_z,[\hat{S}_z,\hat{\rho}]] = \gamma_c\left[2\hat{S}_z\hat{\bar{\rho}}\hat{S}_z - \hat{S}_z^2\hat{\bar{\rho}} - \hat{\bar{\rho}}\hat{S}_z^2\right] \tag{8.106}$$

is the Liouvillean of collisional damping.

8.7 Master Equation for a Three-Level Atom

Consider a three-level atom described by one of the hamiltonians (7.47)-(7.49) depending upon the configuration of the three levels. For the sake of definiteness, let the levels be in the ladder configuration. The hamiltonian of such an atom in a multimode field reads

$$\hat{H} = \sum_{p=g,i,e} E_p\hat{A}_{pp} + \hbar\sum_k \omega_k\hat{a}_k^\dagger\hat{a}_k + \left[\hat{A}_{gi}\hat{F}_1^\dagger + \hat{A}_{ie}\hat{F}_2^\dagger + \mathrm{h.c.}\right] \tag{8.107}$$

where $\hat{F}_1$ ($\hat{F}_2$) are given by (8.50) with g_{1k} (g_{2k}) as the coupling constant between the mode k and the levels $|g\rangle$ and $|i\rangle$ ($|i\rangle$ and $|e\rangle$). The interaction in (8.107) is of the form (8.16) with

$$\hat{S}_1 = \hat{A}_{gi}, \quad \hat{S}_2 = \hat{A}_{ie}, \quad \Omega_1 = \omega_{ig}, \quad \Omega_2 = \omega_{ei}, \tag{8.108}$$

$\omega_{ig} = (E_i - E_g)/\hbar$, $\omega_{ei} = (E_e - E_i)/\hbar$. Assuming the bath to be thermal, the master equation then assumes the form (ignoring frequency shifts)

$$\begin{aligned}\dot{\hat{\rho}} = &-\mathrm{i}\Big[\sum_{k=g,i,e} E_k\hat{A}_{kk},\ \rho\Big] \\ &+ \sum_{i,j=1,2}\gamma_i\Big[(\bar{n}_i+1)\Big(2\hat{S}_j\rho\hat{S}_i^\dagger - \rho\hat{S}_i^\dagger\hat{S}_j - \hat{S}_i^\dagger\hat{S}_j\rho\Big) \\ &+\bar{n}_i\Big(2\hat{S}_j^\dagger\rho\hat{S}_i - \rho\hat{S}_i\hat{S}_j^\dagger - \hat{S}_i\hat{S}_j^\dagger\rho\Big)\Big].\end{aligned} \tag{8.109}$$

Here $\gamma_i \equiv \gamma(\Omega_i)$, $\bar{n}_i \equiv \bar{n}(\Omega_i)$. Now, if $|\omega_{ig} - \omega_{ei}|t \gg 1$ then $i \neq j$ terms in (8.109) can be ignored as, in the interaction picture, those terms contribute factors oscillating at the frequency $|\omega_{ig} - \omega_{ei}|$ which average to zero on the time scale of observation. Hence, the master equation for nearly degenerate sets of level separations will have additional terms as compared with widely separated sets of levels.

8.8 Master Equation for Field Interacting with a Reservoir of Atoms

So far we have considered a system of atoms interacting with a reservoir of the e.m. field. The field frequencies are continuously distributed whereas the atomic system is characterized by a denumerable set of frequencies. Consider now the situation in which the atomic frequencies are almost continuously distributed interacting with field modes described by a set of denumerable frequencies. Such a situation is realized, for example, by well-separated field modes in a cavity. The atoms in the walls of the cavity then act as a reservoir whereas each cavity mode is a small system.

A simple model to describe the atomic oscillators in the walls of a cavity is in terms of harmonic oscillators. The hamiltonian for the combined system of a field mode, described by the operators $\hat{a}$, $\hat{a}^\dagger$, and the atomic oscillators described by the operators $\{\hat{b}_k, \hat{b}_k^\dagger\}$ is then given by (8.80). The corresponding master equation is (8.82) with $\bar{N} \to \bar{n}$, $\bar{M} = 0$ and γ is now the damping constant of the field.

There are, however, situations when the reservoir of atoms can not be modelled as a collection of harmonic oscillators. In the following we derive master equation for the evolution of one or two field modes interacting with a reservoir of two level or effective two level atoms.

- Let a single mode field of frequency ω interact with a reservoir of two-level atoms of frequencies $\omega_1, \omega_2, \cdots$ described by

$$\hat{H} = \hbar\omega_0 \hat{a}^\dagger \hat{a} + \hbar \sum_i \omega_i \hat{S}_z^{(i)} + \hbar \sum_i g_i \hat{S}_+^{(i)} \hat{a} + \text{h.c.} \tag{8.110}$$

The interaction hamiltonian is of the form (8.16) with

$$\hat{S}_1 = \hat{a}, \qquad \hat{F}_1 = \sum_i g_i \hat{S}_-^{(i)}. \tag{8.111}$$

We assume that the atomic reservoir is in a state of thermal equilibrium at temperature T characterized by the density operator

$$\hat{\rho}_{\text{atoms}} = \prod_{i=1} \left[\exp(-\beta_i/2) + \exp(\beta_i/2)\right]^{-1} \exp(-\beta_i \hat{S}_z^{(i)}), \tag{8.112}$$

$\beta_i = \hbar\omega_i / k_B T$. In this state,

$$\langle \hat{S}_\pm^{(i)} \rangle = 0, \quad \langle \hat{S}_z^{(i)} \rangle = -\frac{1}{2(2\bar{n}_i + 1)}, \quad \bar{n}_i = \frac{\exp(-\beta_i)}{1 - \exp(-\beta_i)}. \tag{8.113}$$

We see that in the same approximation as used for deriving (8.59), and on ignoring the principal parts,

$$W_{11}^{(2)} = \frac{(\bar{n}+1)\,\kappa}{(2\bar{n}+1)}, \quad W_{11}^{(1)} = \frac{\bar{n}\kappa}{(2\bar{n}+1)}, \quad W_{11}^{(3)} = W_{11}^{(4)} = 0. \tag{8.114}$$

where $\bar{n} = \bar{n}(\omega_0)$ and κ is the same as $\gamma(\omega_0)$ of (8.60). The summation there is converted in to an integral by applying (8.56) with $h(\omega)$ there being the atomic lineshape function. The master equation for the field mode in question is then the same as (8.82) with $\bar{N}+1 \to (\bar{n}+1)/(2\bar{n}+1)$, $\bar{N} \to \bar{n}/(2\bar{n}+1), \bar{M} = 0$.

- Consider a single-mode field causing two-photon transitions in a system of two-level atoms in the ladder configuration. It is described by (7.51) rewritten in the form

$$\hat{H}_{\rm RS} = \sum_i g_i \hat{S}_+^{(i)} \hat{a}\hat{b} + \text{h.c.} \tag{8.115}$$

This is of the form (8.16) with $\hat{S}_1 = \hat{a}\hat{b}$, and $\hat{F}_1$ is as in (8.111). Assuming the state of the reservoir of atoms to be (8.112), verify that the master equation for the density operator of the field modes is

$$\begin{aligned}\dot{\hat{\rho}} = &-\mathrm{i}\omega_a \left[\hat{a}^\dagger\hat{a}, \hat{\rho}\right] - \mathrm{i}\omega_b \left[\hat{b}^\dagger\hat{b}, \hat{\rho}\right] - \mathrm{i}\eta \left[\hat{a}^\dagger\hat{b}^\dagger\hat{a}\hat{b}, \hat{\rho}\right] \\ &+ \frac{\kappa(\bar{n}+1)}{(2\bar{n}+1)} \left[2\hat{b}\hat{a}\hat{\rho}\hat{a}^\dagger\hat{b}^\dagger - \hat{a}^\dagger\hat{b}^\dagger\hat{a}\hat{b}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger\hat{a}\hat{b}\right] \\ &+ \frac{\kappa\bar{n}}{(2\bar{n}+1)} \left[2\hat{b}^\dagger\hat{a}^\dagger\hat{\rho}\hat{a}\hat{b} - \hat{a}\hat{b}\hat{a}^\dagger\hat{b}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{b}\hat{a}^\dagger\hat{b}^\dagger\right]. \end{aligned} \tag{8.116}$$

The η in the equation above comes from the principal part of the integrals. The equation for the degenerate process is obtained by replacing $\hat{b}$ by $\hat{a}$.

- For the levels in the Lambda configuration, the interaction is governed by (7.52), rewritten in the form

$$\hat{H}_{\rm RS} = \sum_i G_i \hat{S}_+^{(i)} \hat{a}^\dagger\hat{b} + \text{h.c.} \tag{8.117}$$

For the atomic reservoir in the state (8.112) the master equation reads

$$\begin{aligned}\dot{\hat{\rho}} = &-\mathrm{i}\omega_a \left[\hat{a}^\dagger\hat{a}, \hat{\rho}\right] - \mathrm{i}\omega_b \left[\hat{b}^\dagger\hat{b}, \hat{\rho}\right] - \mathrm{i}\eta \left[\hat{a}^\dagger\hat{b}^\dagger\hat{a}\hat{b}, \hat{\rho}\right] \\ &+ \frac{\kappa(\bar{n}+1)}{(2\bar{n}+1)} \left[2\hat{a}^\dagger\hat{b}\hat{\rho}\hat{b}^\dagger\hat{a} - \hat{b}^\dagger\hat{a}\hat{a}^\dagger\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{a}\hat{a}^\dagger\hat{b}\right] \\ &+ \frac{\kappa\bar{n}}{(2\bar{n}+1)} \left[2\hat{b}^\dagger\hat{a}\hat{\rho}\hat{a}^\dagger\hat{b} - \hat{a}^\dagger\hat{b}\hat{b}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{b}\hat{b}^\dagger\hat{a}\right]. \end{aligned} \tag{8.118}$$

The equation for the degenerate process is obtained by replacing $\hat{b}$ by $\hat{a}$.

The method of the solution of these equations is outlined in Chap. 13.

9. Linear and Nonlinear Response of a System in an External Field

We have seen in the preceeding chapters that the problem of studying an optical process reduces to that of solving an appropriate master equation. However, barring the harmonic oscillator model of atoms, the master equations in question can seldom be tackled analytically exactly. This is particularly so when there are more than one frequencies coupling a transition. We identify in Chaps. 12 and 14 analytically exactly solvable atomic systems in which a transition is driven by only one frequency. In this chapter we discuss a perturbative approach to solving the master equation of an atomic system in which a transition may be driven by more than one frequency.

9.1 Steady State of a System in an External Field

Consider a system described by a density operator $\hat{\rho}$ whose evolution is governed by the time-independent master equation

$$\dot{\hat{\rho}}(t) = \hat{\hat{L}}_0 \hat{\rho}(t). \tag{9.1}$$

We assume that the eigenvalues and the eigenvectors of $\hat{\hat{L}}_0$ are known and that one of the eigenvalues of $\hat{\hat{L}}_0$ is zero whereas the real part of all its other eigenvalues is negative. Hence, as $t \to \infty$, the atomic system approaches the steady state $\hat{\rho}_{\rm ss}$ satisfying

$$\hat{\hat{L}}_0 \hat{\rho}_{\rm ss} = 0. \tag{9.2}$$

We assume that the solution of (9.2) is unique. Let the system be subject to an external influence after it has attained the steady state $\hat{\rho}_{\rm ss}$. We wish to study the properties of the system as $t \to \infty$ after the application of an external field.

To that end, let $\hat{H}_{\rm ext}(t)$ be the hamiltonian of interaction between the system and the externally applied field which is treated classically. The density operator $\hat{\rho}$, after the application of the field, evolves according to

$$\dot{\hat{\rho}}(t) = \left[\hat{\hat{L}}_1(t) + \hat{\hat{L}}_0\right] \hat{\rho}(t), \qquad \hat{\hat{L}}_1(t)\hat{\rho} = -\frac{\mathrm{i}}{\hbar}\left[\hat{H}_{\rm ext}(t),\ \hat{\rho}\right]. \tag{9.3}$$

Let the external field be such that $||\hat{\hat{L}}_1|| \ll \hat{\hat{L}}_0$. Hence, (9.3) may be solved perturbatively in powers of $\hat{\hat{L}}_1(t)$. This task is facilitated by applying (1.51) to rewrite the formal solution of (9.3) as

$$\begin{aligned}\hat{\rho}(t) &= \exp\left(\hat{\hat{L}}_0 t\right) \overleftarrow{T} \exp\left[\int_0^t \mathrm{d}\tau \hat{\hat{L}}_I(\tau)\right] \hat{\rho}_{\mathrm{ss}} \\ &\equiv \left[\hat{\rho}_{\mathrm{ss}} + \sum_{n=1}^{\infty} \hat{\rho}^{(n)}(t)\right], \end{aligned} \tag{9.4}$$

$$\hat{\hat{L}}_I(t) = \exp\left(-\hat{\hat{L}}_0 t\right) \hat{\hat{L}}_1(t) \exp\left(\hat{\hat{L}}_0 t\right). \tag{9.5}$$

The $\hat{\rho}^{(n)}(t)$ in (9.4) is the n^{th} order perturbative contribution to the density operator. On carrying the time-ordered expansion in the first line in (9.4) and on using (9.5) and (9.2), we find that

$$\hat{\rho}^{(n)}(t) = \overleftarrow{T} \prod_{i=1}^{n} \left[\int_0^{\tau_{i+1}} \mathrm{d}\tau_i \exp\left\{(\tau_{i+1} - \tau_i)\, \hat{\hat{L}}_0\right\} \hat{\hat{L}}_1(\tau_i)\right] \hat{\rho}_{\mathrm{ss}}, \tag{9.6}$$

with $\tau_{n+1} = t \geq \tau_{n-1} > \cdots \geq \tau_1$. On transforming the integration variables successively as $t - \tau_n \to \tau_n, t - \tau_n - \tau_{n-1} \to \tau_{n-1}, \cdots$, and on letting $t \to \infty$, (9.6) reduces to

$$\hat{\rho}^{(n)}(\infty) = \overleftarrow{T} \prod_{i=1}^{n} \int_0^{\infty} \mathrm{d}\tau_i \exp(\hat{\hat{L}}_0 \tau_i) \hat{\hat{L}}_1 \left(t - \sum_{j=i}^{n} \tau_j\right) \hat{\rho}_{\mathrm{ss}}. \tag{9.7}$$

Now, let the external field be a linear combination of M quasimonochromatic fields whose frequencies are centered at $\nu_1, \ldots, \nu_M$. Let the field-system coupling be described by

$$\hat{H}_{\mathrm{ext}}(t) = \hbar \sum_{\lambda} \sum_{j=-M}^{M} \varepsilon_\lambda(\nu_j, t) \exp(\mathrm{i}\nu_j t) \hat{Q}_\lambda(\nu_j), \tag{9.8}$$

$\nu_{-j} = -\nu_j$, $\varepsilon_\lambda(-\nu_j, t) = \varepsilon_\lambda^*(\nu_j, t)$ are c-numbers, and $\hat{Q}_\lambda(\nu_j)$ is a system operator such that $\hat{Q}_\lambda(\nu_j) = \hat{Q}_\lambda^\dagger(-\nu_j)$. On inserting (9.8) in the definition (9.3) of $\hat{\hat{L}}_1$ and substituting it, in turn, in (9.7) we obtain

$$\begin{aligned}\rho^{(n)}(\infty) = \sum_{\{\lambda_k\}} \sum_{\{i_k=-M\}}^{M} & \exp\left\{\mathrm{i}\left(\nu_{i_1} + \cdots + \nu_{i_n}\right)t\right\} \\ & \times \overleftarrow{T} \prod_{j=1}^{n} \int_0^{\infty} \mathrm{d}\tau_j \left[\varepsilon_{\lambda_j}\left(\nu_{i_j},\ t - \sum_{k=j}^{n} \tau_k\right)\right. \\ & \left. \times \exp\left\{\hat{\hat{L}}_0 \tau_j - \mathrm{i}\nu_{i_j} \sum_{k=j}^{n} \tau_k\right\} \hat{\hat{L}}_{\lambda_j}\left(\nu_{i_j}\right)\right] \hat{\rho}_{\mathrm{ss}}, \end{aligned} \tag{9.9}$$

$$\hat{\hat{L}}_\lambda(\nu)\hat{\rho} = -\mathrm{i}\left[\hat{Q}_\lambda(\nu),\ \hat{\rho}\right]. \tag{9.10}$$

Let each of the component in (9.8) be perfectly monochromatic so that $\varepsilon_\lambda(\nu_i, t)$ are independent of time and let $\varepsilon_\lambda(\nu_i, t) \to \varepsilon_\lambda(\nu_i)$. The integration in (9.9) can then be performed formally to give

$$\rho^{(n)}(\infty) = \sum_{\{i_k=-M\}}^{M} \sum_{\{\lambda_k\}} \exp\left\{\mathrm{i}\left(\nu_{i_1} + \cdots + \nu_{i_n}\right)t\right\} \overleftarrow{\prod_{j=1}^{n}} \varepsilon_{\lambda_j}(\nu_{i_j})$$
$$\times \left[\hat{\hat{L}}_0 - \mathrm{i}\sum_{k=1}^{j} \nu_{i_k}\right]^{-1} \hat{\hat{L}}_{\lambda_j}\left(\nu_{i_j}\right)\hat{\rho}_{\mathrm{ss}}. \tag{9.11}$$

The left arrow on the product denotes that the product index number increases from right to left. Note that the exponential term and the product of the field amplitudes in (9.11) are invariant under the exchange of the sets (λ_j, ν_{i_j}) and (λ_k, ν_{i_k}) $(i, j = 1, \ldots, n)$ but not the operator part in it. Hence, the contribution to a particular value of $\nu_{i_1} + \cdots + \nu_{i_n}$ and a particular product of the field amplitudes to (9.11) arises from a sum of the operators in it obtained by exchange of n sets of indices $(\lambda_1, \nu_{i_1}), \cdots, (\lambda_n, \nu_{i_n})$. Keeping this in mind, the n^{th} order contribution $A^{(n)} \equiv \mathrm{Tr}(\hat{A}\rho^{(n)}(\infty))$ to the expectation value $\mathrm{Tr}(\hat{A}\rho)$ of a system operator $\hat{A}$ may be expressed as

$$A^{(n)} = \sum_{\{i_k=-M\}}^{M} \Big[\exp\left(\mathrm{i}(\nu_{i_1} + \cdots + \nu_{i_n})t\right)$$
$$\times S_{\mathrm{A}\lambda_1,\cdots,\lambda_n}\left(\nu_{i_1}, \cdots, \nu_{i_n}\right) \prod_{j=1}^{n} \varepsilon_{\lambda_j}(\nu_{i_j})\Big], \tag{9.12}$$

where the so called *response function* of the operator $\hat{A}$ is defined by

$$S_{\mathrm{A}\lambda_1,\cdots,\lambda_n}\left(\nu_{i_1}, \cdots, \nu_{i_n}\right) = \frac{1}{n!}\mathrm{Sym}\left[\mathrm{Tr}\left\{\hat{A}\left(\hat{\hat{L}}_0 - \mathrm{i}\sum_{k=1}^{n} \nu_{i_k}\right)^{-1} \hat{\hat{L}}_{\lambda_n}(\nu_{i_n})\right.\right.$$
$$\left.\left. \times \cdots \left(\hat{\hat{L}}_0 - \mathrm{i}\nu_{i_1}\right)^{-1} \hat{\hat{L}}_{\lambda_1}(\nu_{i_1})\hat{\rho}_{\mathrm{ss}}\right\}\right], \tag{9.13}$$

with Sym standing for the operation of symmetrization of the product on its right in the indices $(\lambda_1, \nu_{i_1}), \cdots, (\lambda_n, \nu_{i_n})$. If $\hat{A} = \hat{Q}_\lambda$ then the response function is called a *susceptibility* of the system.

We apply the results of this section to the problem of the response of an atomic system to an externally applied e.m. field.

9.2 Optical Susceptibility

Let us express the hamiltonian of interaction between an atom and the field in the form

$$\hat{H}_{\rm ext}(t) = \hbar \sum_{\lambda} \sum_{i=-M}^{M} \hat{d}_{\lambda}(\nu_j)\varepsilon_{\lambda}(\nu_j)\exp(\mathrm{i}\nu_i t). \tag{9.14}$$

If the interaction is without the RWA, then $\hat{d}_{\lambda}(\nu) = \hat{d}_{\lambda}$ where $\hat{d}_{\lambda} = \hat{\boldsymbol{d}} \cdot \boldsymbol{\epsilon}_{\lambda}$ is the component of the atomic dipole operator in the polarization direction $\boldsymbol{\varepsilon}_{\lambda}$ of the field. Recall that

$$\hat{d} = \sum_{i,j=1}^{N} d_{ij}|i\rangle\langle j|, \tag{9.15}$$

where $|i\rangle$ $(i = 1, 2, \ldots, N)$ is an atomic state of energy E_i and d_{ij} is the electric dipole moment component between the states $|i\rangle$ and $|j\rangle$ in the direction $\boldsymbol{\epsilon}$. In the RWA, and for $\nu > 0$,

$$\hat{d}(\nu) = \sum_{E_i<E_j} d_{ij}|i\rangle\langle j|, \qquad \hat{d}(-\nu) = \sum_{E_i>E_j} d_{ij}^{*}|i\rangle\langle j|, \tag{9.16}$$

The label ν on $\hat{d}_{\lambda}(\nu)$ thus enables us to incorporate compactly the RWA in the formalism.

The hamiltonian (9.14) is in the form (9.8). The $n^{\rm th}$ order contribution to $\langle \hat{d}_{\lambda}(-\nu)\rangle$ in the asymptotic limit $t \to \infty$ is, therefore, given by (9.12), with $\hat{A} \to \hat{d}_{\lambda}(-\nu)$:

$$\begin{aligned} d_{\lambda}^{(n)}(-\nu) = & \sum_{i_k=-M}^{M} \exp\left(\mathrm{i}(\nu_{i_1} + \cdots + \nu_{i_n})t\right) \\ & \times \sum_{\{\lambda_i\}} \chi^{(n)}_{\lambda\lambda_1,\cdots,\lambda_n}(\nu_{i_1}, \cdots, \nu_{i_n}\}) \prod_{j=1}^{n} \varepsilon_{\lambda_j}(\nu_{i_j}), \end{aligned} \tag{9.17}$$

where

$$\begin{aligned} & \chi^{(n)}_{\lambda\lambda_1,\cdots,\lambda_n}(\nu_{i_1}, \cdots, \nu_{i_n}) \\ & = \frac{1}{n!}\mathrm{Sym}\Big[\mathrm{Tr}\Big\{\hat{d}_{\lambda}(-\nu)\left(\hat{\hat{L}}_0 - \mathrm{i}\sum_{k=1}^{n}\nu_{i_k}\right)^{-1}\hat{\hat{L}}_{\lambda_n}(\nu_{i_n}) \\ & \times \cdots \left(\hat{\hat{L}}_0 - \mathrm{i}\nu_{i_1}\right)^{-1}\hat{\hat{L}}_{\lambda_1}(\nu_{i_1})\hat{\rho}_{\rm ss}\Big\}\Big] \end{aligned} \tag{9.18}$$

is the *optical susceptibility* of the atomic medium and

$$\hat{\hat{L}}_{\lambda}(\nu)\hat{\rho} = -\frac{\mathrm{i}}{\hbar}\left[\hat{d}_{\lambda}(\nu),\ \hat{\rho}\right]. \tag{9.19}$$

If $n_{\rm d}$ is the density per unit volume of the atoms then the dipole moment per unit volume of such a collection of atoms is given by $n_{\rm d}\langle\hat{d}\rangle$.

Notice that if the system is symmetric under space inversion then, due to the fact that the dipole operator is odd under space inversion, it follows that

$$\chi^{(2n)}_{\lambda,\lambda_1,\cdots,\lambda_{2n}}(\nu_{i_1}, \cdots, \nu_{i_{2n}}) = 0, \tag{9.20}$$

i.e. even-order susceptibilities vanish if the system is space-inversion symmetric. This implies in particualr that even-order susceptibilities of a free atomic system are zero. Details of the relationships between susceptibilites arising as a result of the symmetries of the system may be found, for example, in [96, 97].

Further simplification of the expression (9.18) for the susceptibility is achieved by noting that $|\varepsilon|$ is usually small compared with the optical frequencies. As a result, we may ignore those denominators in (9.18) which contain terms of the kind $\Gamma + \mathrm{i}(\omega + \Omega)$, $(\Omega > 0)$. It leads essentially to the same results as are obtained by making the rotating wave approximation (RWA) on the hamiltonian along with the use of (9.16).

The nth order susceptibility determines the contribution to the dipole moment induced by the applied fields in the n^{th} order of perturbation. The induced moment is linear in the amplitudes of the applied fields in the first order and non-linear in the higher orders. The induced moment oscillates at linear combinations of the applied frequencies. The n^{th} order contribution to the dipole moment consists of all the combinations of the n $\nu_i's$ selected from the set of applied frequencies $\nu_{\pm 1}, \nu_{\pm 2}, \cdots, \nu_{\pm M}$. The oscillations in the first order are at the frequencies of the applied fields whereas new oscillation frequencies appear in the higher orders of the perturbation due to linear algebraic combination of frequencies. This is the process of wave mixing. Note that a particular frequency may arise not in one but in several orders of perturbation. The dominant contribution to a particular frequency comes from the lowest order in perturbation in which it appears. Let n be the lowest order of perturbation in which a combination Ω_n of the frequencies appears. Let $d^{(n)}(\Omega_n)$ be the lowest order component of the dipole oscillating at Ω_n. The radiation at frequency Ω_n may be visualized as coming from a dipole of moment $d^{(n)}(\Omega_n)$. Note that the field in the radiation zone from an oscillator is given by (8.76). It shows that the intensity of field from an oscillator of dipole moment $\boldsymbol{d}$ is proportional to $|\boldsymbol{d}|^2$. Hence the intensity of the field at Ω_n produced in the wave mixing is proportional to $|d^{(n)}(\Omega_n)|^2$. Since $d(\Omega_n)$ is proportional to $\chi^{(n)}_{\lambda\lambda_1,\cdots,\lambda_n}(\nu_{i_1}, \cdots, \nu_{i_n})$ where $\nu_{i_1} + \cdots + \nu_{i_n} = \Omega_n$, the intensity of the radiation of frequency Ω_n from a driven atomic system is proportional to $|\chi^{(n)}_{\lambda\lambda_1,\cdots,\lambda_n}(\nu_{i_1}, \cdots, \nu_{i_n})|^2$.

We may write explicit expression for susceptibilities of various wave-mixing processes. We have derived such an expression in Chap. 12 for investigating the process of four-wave mixing in a bichromatic field. For the present, we derive in the next section the rate of absorption of energy by the system from the field in the first order.

9.3 Rate of Absorption of Energy

Consider a system subject to an external field of frequency ν so that the interaction hamiltonian is given by (9.8) with $M = 1$ and $\nu_1 \equiv \nu$. Now, the

rate of change of the mean internal energy $U(t)$ of a system described by the hamiltonian $\hat{H}$ is given by [96]

$$W_{\rm a}(\nu) = \frac{{\rm d}U}{{\rm d}t} = \frac{\partial\langle\hat{H}\rangle}{\partial t}. \tag{9.21}$$

In the present case, the explicit time-dependence of the hamiltonian arises only from the externally applied field. It then follows that

$$W_{\rm a} = {\rm i}\nu \sum_{\lambda} \left[\varepsilon_\lambda(\nu)\langle\hat{Q}_\lambda(\nu)\rangle \exp({\rm i}\nu t) - {\rm c.c.}\right]. \tag{9.22}$$

Substitute for the average in the equation above the expression (9.11) to the first-order. Note that the resulting expression contains exponentials like $\exp(\pm{\rm i}\nu t)$, $\exp(\pm{\rm i}2\nu t)$ and the terms without any t-dependence. If the time of observation is very long compared with ν^{-1} then the observed values are averages over several periods of oscillations. Such an average of the oscillating terms gives vanishing contribution. Hence, keeping only the time-independent terms, we get

$$\begin{aligned} W_{\rm a} &= \nu \sum_{\lambda,\beta} \varepsilon_\lambda \varepsilon_\beta^* {\rm Tr}\left\{\hat{Q}_\lambda \left(\hat{\hat{L}}_0 + {\rm i}\nu\right)^{-1} \left[Q_\beta(-\nu),\ \hat{\rho}_{\rm ss}\right]\right\} + {\rm c.c.} \\ &= \nu \int_0^\infty {\rm d}\tau \sum_{\lambda,\beta} \varepsilon_\lambda \varepsilon_\beta^* {\rm Tr}\left\{\hat{Q}_\lambda \exp\left((\hat{\hat{L}}_0 + {\rm i}\nu)\tau\right)\left[Q_\beta(-\nu),\ \hat{\rho}_{\rm ss}\right]\right\} + {\rm c.c.} \\ &= \nu \sum_{\lambda,\beta} \varepsilon_\lambda \varepsilon_\beta^* \int_0^\infty {\rm d}\tau \exp({\rm i}\nu\tau) \left\langle \left[\hat{Q}_\lambda(\tau,\nu),\ Q_\beta(-\nu)\right]\right\rangle + {\rm h.c.}, \end{aligned} \tag{9.23}$$

$$\hat{Q}(t) = \exp\left(\hat{\hat{L}}_0 t\right)\hat{Q}. \tag{9.24}$$

This determines the rate at which the system absorbs energy from the field in terms of the two-time correlation function of the system operators. Variation of $W_{\rm a}(\nu)$ as a function of ν gives the *absorption spectrum* of the field. The first equation in (9.23) shows that the resonances in the absorption spectrum are located at the imaginary parts of the eigenvalues of $\hat{\hat{L}}_0$.

For the case of a two-level atom in a single component monochromatic field, the hamiltonian in the RWA is given by (9.14) with $M = 1$, $\lambda = 1$, and by virtue of (9.16),

$$\hat{d}(\nu) = d_{ge}|g\rangle\langle e| \equiv d_{ge}\hat{S}_-, \qquad \hat{d}(-\nu) = d_{ge}^*\hat{S}_+. \tag{9.25}$$

The expression (9.23) for the absorption spectrum then reads

$$\begin{aligned} W_{\rm a}(\nu) &= 2\nu|\varepsilon|^2 {\rm Re}\int_0^\infty d\tau\ \exp({\rm i}\nu\tau)\left\langle\left[\hat{S}_-(\tau), \hat{S}_+\right]\right\rangle. \\ &= 2\nu|\varepsilon|^2 {\rm Re}\left[{\rm Tr}\left\{\hat{S}_-\left(\hat{\hat{L}}_0 + {\rm i}\nu\right)^{-1}\left[\hat{S}_+,\ \hat{\rho}_{\rm ss}\right]\right\}\right]. \end{aligned} \tag{9.26}$$

This determines the absorption characteristics of a two-level atom.

9.4 Response in a Fluctuating Field

We have thus far confined our attention to the asymptotic response of a system to a linear combination of discretely spaced monochromatic fields, though we have at hand also the expression (eq.(9.9)) for the density operator when the fields are quasimonochromatic. A situation of considerable interest concerns the fields fluctuating around a mean frequency. The expectation value of an observable $\hat{A}$ in the n^{th} order in this case is found by first evaluating $\text{Tr}[\hat{A}\hat{\rho}^{(n)}(\infty)]$ and then averaging it over the fluctuations:

$$\overline{A^{(n)}} = \overline{\text{Tr}\left[\hat{A}\hat{\rho}^{(n)}(\infty)\right]}. \tag{9.27}$$

The bar denoting average over the fluctuations. This may be evaluated by finding $\overline{\rho^{(n)}(\infty)}$. Now, consider the signal $S(\nu)$ at the frequency ν resulting from mixing of the applied frequencie. If domonant contribution to $S(\nu)$ arises in the n^{th} order then the signal averaged over the field fluctuations is

$$\overline{S(\nu)} \sim \overline{|d^{(n)}(\nu)|^2}, \tag{9.28}$$

where $d^{(n)}(\nu)$ is the average of $\hat{d}(\nu)$ found by applying (9.27). Clearly, $S(\nu)$ is not linear in $\hat{\rho}^{(n)}$ and hence the signal average over fluctuations can not be derived by averaging the density operator over fluctuations. Carrying the average in (9.28) for non-linear response is generally an involved task. Calculations for the second and the third order response for some models of fluctuations may be found in [98]. The reference [99] draws attention to erroneous conclusions arrived at by first averaging $d^{(n)}$ over fluctuations and then squaring it.

10. Solution of Linear Equations: Method of Eigenvector Expansion

We have seen in previous chapters that a variety of dynamical problems reduce to solving a set of coupled linear first order equations expressible in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = \hat{X}|\psi(t)\rangle, \tag{10.1}$$

where $|\psi(t)\rangle$ is a vector in an n-dimensional space and $\hat{X}$ an operator acting on the vectors in that space. The operator $\hat{X}$ may or may not be time-dependent. In this chapter we assume $\hat{X}$ to be independent of time. The formal solution of (10.1) then is

$$|\psi(t)\rangle = \exp(\hat{X}t)|\psi(0)\rangle. \tag{10.2}$$

In principle, we can evaluate (10.2) by expanding the exponential in powers of $\hat{X}$ and by evaluating $\hat{X}^n|\psi(0)\rangle$. This procedure may be simplified by expressing the exponential as products of exponentials of operators whose action on $|\psi(0)\rangle$ is simpler to evaluate. The problem of disentangling an exponential has been addressed in Chap. 2. Alternatively, we may express $|\psi(0)\rangle$ in a basis which is such that the action of $\hat{X}$ on a basis state results in a linear combination of fewer number of vectors than the dimension n of the space in question. In other words, we would like to choose the basis vectors which are reducible to a sum of subspaces each of which is invariant under the action of $\hat{X}$. Smaller the dimension of such subspaces to which a basis can be reduced, simpler it is to handle. The most desirable choice then is the basis which can be reduced to a sum of one-dimensional invariant subspaces. A one-dimensional subspace invariant under the action of an operator is constituted by the eigenvector of the operator. Hence, the set of all the eigenvectors of $\hat{X}$ is the most desirable basis for evaluating (10.2). However, the set of eigenvectors of an arbitrary $\hat{X}$ need not be complete. We then have to discover additional vectors to make the set complete. These additional vectors are constructed by introducing the concept of generalized eigenvectors. In this chapter we discuss the eigenvalue problem of an operator to evaluate (10.2).

10.1 Eigenvalues and Eigenvectors

Recall from (1.32) that λ is an eigenvalue of $\hat{X}$ and $|\psi_\lambda\rangle$ the corresponding eigenvector if

$$\left[\hat{X} - \lambda I\right]|\psi_\lambda\rangle = 0. \tag{10.3}$$

If $[\hat{X} - \lambda\hat{I}]^{-1}$ exists, then we can operate (10.3) on the left with $[\hat{X} - \lambda\hat{I}]^{-1}$ leading to the conclusion that $|\psi_\lambda\rangle = 0$. Thus (10.3) admits non-trivial solution only for those values of λ for which $\hat{X} - \lambda\hat{I}$ is singular.

Recall from Chap. 1 that a vector in an n-dimensional space may be represented by a column of n rows and an operator by an $n \times n$ matrix. We recall also the theorem that necessary and sufficient condition for a finite-dimensional matrix $\hat{A}$ to be singular is that $\det(\hat{A})$ be zero [100]. Hence, the necessary and sufficient condition for (10.3) to admit a non-trivial solution is

$$\det(\hat{X} - \lambda I) = 0. \tag{10.4}$$

This equation determines the eigenvalues λ. In the following we enumerate some properties of eigenvalues and eigenvectors. Some results are stated without proof. Their proofs may be found in [3, 100].

1. The equation(10.4) is a polynomial of degree n and hence admits n roots $\lambda_1, \cdots, \lambda_n$. However, not all the roots need be distinct. The number of times a root is repeated is called its *multiplicity*. Let $\lambda_1, \cdots, \lambda_m$ be m distinct roots. Let r_i be the multiplicity of λ_i so that (10.4) may be written as

$$(\lambda - \lambda_1)^{r_1} \cdots (\lambda - \lambda_m)^{r_m} = 0, \quad \sum_{i=1}^{m} r_i = n. \tag{10.5}$$

   Corresponding to each λ_i of multiplicity r_i, (10.3) admits a number $n_i \leq r_i$ of non-trivial solutions. The eigenvector corresponding to an eigenvalue of multipicity one is unique. The multiplicity of a root is also called the *geometrical degeneracy* of the eigenvalue and the number of independent eigenvectors corresponding to a multiple root its *dynamical degeneracy*. Since a polynomial has at least one root, (10.3) admits at least one non-trivial solution.
2. The eigenvalues of the adjoint of an operator are complex conjugates of the eigenvalues of the operator. To prove this, let

$$\hat{X}^\dagger|\phi_\mu\rangle = \mu|\phi_\mu\rangle. \tag{10.6}$$

   This implies

$$\langle\phi_\mu|\hat{X} = \mu^*\langle\phi_\mu|. \tag{10.7}$$

   The eigenvalue μ of $\hat{X}^\dagger$ is a solution of $\det(\hat{X}^\dagger - \mu I) = 0$. Since $\det(\hat{A}^\dagger) = (\det(\hat{A}))^*$, it follows that $\mu = \lambda^*$ where λ is an eigenvalue of $\hat{X}$.

3. Property 2 implies that the eigenvalues of a hermitian operator are real.
4. The eigenvectors of $\hat{X}$ corresponding to an eigenvalue λ are orthogonal to the eigenvectors of $\hat{X}^\dagger$ corresponding to the eigenvalues $\mu \neq \lambda^*$. To prove it, take the scalar product of the eigenvalue equation (10.3) for $\hat{X}$ with the eigenvector $|\phi_\mu\rangle$ of $\hat{X}^\dagger$ to get $\langle\phi_\mu|\hat{X} - \lambda I|\psi_\lambda\rangle = 0$. On applying (10.7) it reduces to $(\mu^* - \lambda)\langle\phi_\mu|\psi_\lambda\rangle = 0$. This implies that $\langle\phi_\mu|\psi_\lambda\rangle = 0$ if $\lambda \neq \mu^*$.
5. The eigenvectors corresponding to distinct eigenvalues are linearly independent. To see this, operate the equation $a_1|\psi_1\rangle + \cdots + a_m|\psi_m\rangle = 0$ involving linear combination of m eigenvectors of $\hat{X}$ belonging to distinct eigenvalues $\lambda_1, \cdots, \lambda_m$ by $(\hat{X} - \lambda_1)\cdots(\hat{X} - \lambda_{i-1})(\hat{X} - \lambda_{i+1})\cdots(\hat{X} - \lambda_m)$. It leads to the equation $a_i|\psi_i\rangle = 0$ which implies $a_i = 0$ for all i. Hence, by the definition of linear independence, $|\psi_1\rangle, \cdots, |\psi_m\rangle$ are linearly independent.
6. We infer from 5 that if the eigenvalues of an n-dimensional operator $\hat{X}$ are distinct then it would admit n linearly independent eigenvectors. These eigenvectors may be employed to serve as a basis for the vector space of $\hat{X}$. The set of vectors orthonormal to the set of eigenvectors of $\hat{X}$ is constituted, in view of 4, by the eigenvectors of $\hat{X}^\dagger$. If $|\psi_{\lambda_1}\rangle, \cdots, |\psi_{\lambda_n}\rangle$ are the eigenvectors of $\hat{X}$ corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_n$, and $|\phi_{\lambda_1}\rangle, \cdots, |\phi_{\lambda_n}\rangle$ are the eigenvectors of $\hat{X}^\dagger$ corresponding to the eigenvalues $\lambda_1^*, \ldots, \lambda_n^*$ then the orthonormality and completeness relations involving these vectors read

$$\langle\phi_{\lambda_i}|\psi_{\lambda_j}\rangle = \delta_{ij}, \tag{10.8}$$

$$\sum_{i=1}^{n} |\psi_{\lambda_i}\rangle\langle\phi_{\lambda_i}| = 1. \tag{10.9}$$

Operate (10.9) with $f(\hat{X})$ and invoke (1.33) to obtain

$$f(\hat{X}) = \sum_{i=1}^{n} f(\lambda_i)|\psi_{\lambda_i}\rangle\langle\phi_{\lambda_i}|. \tag{10.10}$$

By setting $f(\hat{X}) = (\hat{X} - \lambda_1)\cdots(\hat{X} - \lambda_{i-1})(\hat{X} - \lambda_{i+1})\cdots(\hat{X} - \lambda_n)$ it follows that

$$|\psi_{\lambda_i}\rangle\langle\phi_{\lambda_i}| = \prod_{j\neq i}^{n} \frac{\hat{X} - \lambda_j}{\lambda_i - \lambda_j}. \tag{10.11}$$

On combining (10.10) and (10.11) we obtain

$$f(\hat{X}) = \sum_{i=1}^{n} f(\lambda_i) \prod_{j\neq i}^{n} \frac{\hat{X} - \lambda_j}{\lambda_i - \lambda_j}. \tag{10.12}$$

We exploit this result in the Appendix C to write general solution of a set of two and three coupled linear equations.

7. An operator acting in an n-dimensional space may admit n linearly independent eigenvectors even when its eigenvalues are not all distinct. That would be the case if the number of independent eigenvectors corresponding to a multiple root is equal to its multiplicity. There is, however, no general prescription for ascertaining the number of independent eigenvectors corresponding to a multiple eigenvalue except in the special case of normal operators defined in Chap. 1. It can be proved that the number of independent eigenvectors corresponding to an eigenvalue of a normal operator is equal to the multiplicity of the eigenvalue [100]. Hence, a normal operator in an n-dimensional space admits n lineraly independent eigenvectors. These eigenvectors may, furthermore, be orthonormalized. More generally, it can be proved that a necessary and sufficient condition for an n-dimensional operator to admit n orthonormal eigenvectors is that it be normal. Recall that the hermitian and unitary operators are special cases of a normal operator. Hence the eigenvectors of these operators constitute an orthonormal basis.
8. Let $|\psi_{\lambda_1}\rangle, \cdots, |\psi_{\lambda_n}\rangle$ and $|\phi_{\lambda_1}\rangle, \cdots, |\phi_{\lambda_n}\rangle$ be as in the item 6 above. Let $\hat{\Psi}$ be a matrix constituted by the column vectors $|\psi_{\lambda_1}\rangle, \cdots, |\psi_{\lambda_n}\rangle$, and $\hat{\Phi}$ that constituted by the column vectors $|\phi_{\lambda_1}\rangle, \cdots, |\phi_{\lambda_n}\rangle$. Using the orthonormality relation (10.7), it is straightforward to show that

$$\hat{\Phi}^{\dagger}\hat{X}\hat{\Psi} = \hat{\Psi}^{-1}\hat{X}\hat{\Psi} = \hat{D}_X(\lambda_1, \ldots, \lambda_n), \tag{10.13}$$

$\hat{D}_X(\lambda_1, \ldots, \lambda_n)$ being a diagonal matrix with the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $\hat{X}$ as its diagonal elements. This shows that an n-dimensional operator admitting n linearly independent eigenvectors can be diagonalized by a similarity transformation. The transformation in question is generated by the matrix formed by the eigenvectors of the operator as columns.

Now, consider the formal solution (10.2) of (10.1). Let $\hat{X}$ admit n linearly independent eigenvectors $|\psi_{\lambda_1}\rangle, \cdots, |\psi_{\lambda_1}\rangle$ corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n$. Using (10.9) we represent $|\psi(0)\rangle$ as

$$|\psi(0)\rangle = \sum_{i=1}^{n} \langle\phi_{\lambda_i}|\psi(0)\rangle|\psi_{\lambda_i}\rangle. \tag{10.14}$$

On substituting this in (10.2) we obtain

$$|\psi(t)\rangle = \sum_{i=1}^{n} \exp(\lambda_i t)\langle\phi_{\lambda_i}|\psi(0)\rangle|\psi_{\lambda_i}\rangle. \tag{10.15}$$

This is the solution of (10.1) in case $\hat{X}$ admits n independent eigenvectors.

The eigenvectors can not constitute a basis if there is any root not having as many independent eigenvectors as its multiplicity. The vectors required in addition to the eigenvectors to make a complete set are then obtained by invoking the concept of a generalized eigenvector introduced next.

10.2 Generalized Eigenvalues and Eigenvectors

Consider an operator $\hat{X}$ in an n-dimensional vector space. Let us assume that it has m distinct eigenvalues, denoted by $\lambda_1, \cdots, \lambda_m$ and that r_i is the multiplicity of λ_i. Let us assume that corresponding to an eigenvalue λ_i there is only one eigenvector, denoted by $|\psi_{\lambda_i}(1)\rangle$, so that

$$\left[\hat{X} - \lambda_i I\right] |\psi_{\lambda_i}(1)\rangle = 0. \tag{10.16}$$

If $r_i \geq 2$ then construct $|\psi_{\lambda_i}(k)\rangle$ by solving

$$\left[\hat{X} - \lambda_i I\right] |\psi_{\lambda_i}(k)\rangle = |\psi_{\lambda_i}(k-1)\rangle \tag{10.17}$$

successively for $k = 2, \ldots, r_i$. On operating (10.17) by $\hat{X} - \lambda_i I$ successively for $k = 1, \ldots$ and on using (10.16) it follows that

$$\left[\hat{X} - \lambda_i I\right]^k |\psi_{\lambda_i}(k)\rangle = 0, \quad \left[\hat{X} - \lambda_i I\right]^{k-1} |\psi_{\lambda_i}(k)\rangle \neq 0. \tag{10.18}$$

Vector $|\psi_{\lambda_i}(k)\rangle$ is called a *generalized eigenvector of rank k* and λ_i a *generalized eigenvalue*. The eigenvector of rank 1, $|\psi_{\lambda_i}(1)\rangle$, is also called the *ordinary eigenvector*. Note that each of the generalized eigenvectors is arbitrary to the addition of a scalar multiple of the ordinary eigenvector. The importance of the concept of generalized eigenvectors stems from their following properties:

1. If the eigenvalue λ_i of multiplicity r_i has only one ordinary eigenvector, then its r_i generalized eigenvectors, $|\psi_{\lambda_i}(k)\rangle$ $(k = 1, \ldots, r_i)$, are linearly independent.
2. Generalized eigenvectors corresponding to different eigenvalues are linearly independent.
3. The set of generalized eigenvectors of $\hat{X}$ is orthonormal to the set of generalized eigenvectors of $\hat{X}^\dagger$. Let Ψ be an $n \times n$ matrix formed by the generalized eigenvectors $|\psi_{\lambda_1}(1)\rangle, \cdots, |\psi_{\lambda_1}(r_1)\rangle; \cdots; |\psi_{\lambda_m}(1)\rangle, \cdots, |\psi_{\lambda_m}(r_m)\rangle$ of $\hat{X}$ as columns numbered 1 to n. Let the matrix Φ be formed similarly by the generalized eigenvectors $|\phi_{\lambda_i}(k)\rangle, \cdots, |\phi_{\lambda_i}(r_i)\rangle$ $(i = 1, \ldots, m)$ of $\hat{X}^\dagger$ corresponding to the eigenvalues $\lambda_1^*, \ldots, \lambda_m^*$. The orthogonality of the generalized eigenvectors of $\hat{X}$ and $\hat{X}^\dagger$ implies that

   $$\hat{\Phi}^\dagger \hat{\Psi} = I. \tag{10.19}$$

   The definition of the generalized eigenvectors and (10.19), imply that

   $$\hat{\Phi}^\dagger \hat{X} \hat{\Psi} = \hat{\Psi}^{-1} \hat{X} \hat{\Psi} = \hat{J}_X, \tag{10.20}$$

   $\hat{J}_X$ being the *Jordan canonical form* of $\hat{X}$. It is such that (i) its elements along the main diagonal are the eigenvalues of $\hat{X}$, (ii) all its elements below the main diagonal are zero, (iii) besides the main diagonal, its only non-zero elements, if any, are in the diagonal above the main diagonal. Each of those non-zero elements is unity. The eigenvalue λ_1 appears in

the main diagonal in the rows numbered 1 to r_1 followed by λ_2 which appears in rows numbered $r_1 + 1$ to $r_1 + r_2$ and so on. Unity appears in the diagonal above the main diagonal in rows numbered 1 to $r_1 - 1$, $r_1 + 1$ to $r_1 + r_2 - 1$ and so on.

We have thus at hand n independent vectors in an n-dimensional space. We employ it as a basis to evaluate (10.2) by expressing $|\psi(0)\rangle$ as

$$|\psi(0)\rangle = \sum_{i=1}^{m} \sum_{k=1}^{r_i} \langle \phi_{\lambda_i}(k)|\psi(0)\rangle |\psi_{\lambda_i}(k)\rangle. \tag{10.21}$$

Substitute this in (10.2) and invoke (10.16)-(10.18) to show that

$$\begin{aligned} \exp(\hat{X}t)|\psi_{\lambda_i}(k)\rangle &= \exp(\lambda_i t) \exp\left[\left(\hat{X} - \lambda_i\right) t\right] |\psi_{\lambda_i}(k)\rangle \\ &= \exp(\lambda_i t) \sum_{l=0}^{k-1} \frac{t^l}{l!} |\psi_{\lambda_i}(k-l)\rangle. \end{aligned} \tag{10.22}$$

As a consequence, we get

$$|\psi(t)\rangle = \sum_{i=1}^{m} \sum_{k=1}^{r_i} \sum_{l=0}^{k-1} \exp(\lambda_i t) \langle \phi_{\lambda_i}(k)|\psi(0)\rangle \frac{t^l}{l!} |\psi_{\lambda_i}(k-l)\rangle. \tag{10.23}$$

This is the solution of (10.1) in case there is only one ordinary eigenvector corresponding to an eigenvalue of any multiplicity. The expression (10.23) shows that the evolution of a state involves terms which are products of an exponential in time with a power of time. Note from (10.15) that the time-dependence is an exponential function if the evolution operator admits as many linearly independent eigenvectors as its dimension.

The problems that we encounter in Chaps. 12 and 13 involving multiple eigenvalues fall in to the category discussed above. We, therefore, do not discuss the general case when an eigenvalue of multiplicity greater than two has two or more independent eigenvectors. In the general case also we can always find r_i independent generalized eigenvectors corresponding to an eigenvalue λ_i of multiplicity r_i. They possess the property (10.18). Since the maximum value of k is r_i, it follows that (10.18) holds for any generalized eigenvector if $k = r_i$. Now, since any vector in the given space is expressible as a linear combination of generalized eigenvectors, it follows that

$$\left(\hat{X} - \lambda_1\right)^{r_1} \cdots \left(\hat{X} - \lambda_m\right)^{r_m} = 0. \tag{10.24}$$

On comparing this with (10.5) we note that an operator satisfies its own eigenvalue equation. This is the content of the *Hamilton–Cayley theorem* [100]. However, (10.24) need not be the minimum polynomial equation satisfied by $\hat{X}$. For, if there are more than one independent eigenvectors corresponding to an eigenvalue λ_i of multiplicity two or more then the expansion (10.21) would contain eigenvectors of lower rank corresponding to λ_i. Consequently, (10.24)

will contain the power of $\hat{X} - \lambda_i$ less than r_i. We have used this property in Sect. 2.2 to express the exponential of a finite-dimensional operator in terms of a polynomial in the operator.

We solve (10.1) in the next section for a special form of $\hat{X}$ without taking recourse to eigenvector expansion. Its eigenvalue equation, along with that of another form, is solved in Sect. 10.4.

10.3 Solution of Two-Term Difference-Differential Equation

In this section we solve the equation

$$\dot{C}_m = \alpha_m C_m + \gamma_m C_{m+1} \qquad m = 0, 1, \ldots, N, \tag{10.25}$$

assuming that $C_{N+1} = 0$, and the equation

$$\dot{C}_m = \alpha_m C_m + \beta_m C_{m-1} \qquad m = 0, 1, \ldots, \tag{10.26}$$

assuming $C_{-1} = 0$.

Consider first (10.25). Its Laplace transform (defined in (8.84)) yields

$$\tilde{C}_m = \frac{1}{z - \alpha_m} \left[C_m(0) + \gamma_m \tilde{C}_{m+1} \right]. \tag{10.27}$$

Set $m = N$ in this and use the given condition $\tilde{C}_{N+1} = 0$ to obtain

$$\tilde{C}_N = \frac{1}{z - \alpha_N} C_N(0). \tag{10.28}$$

Set $m = N-1, N-2, \ldots$, successively in (10.27) to get

$$\tilde{C}_m = \frac{1}{z - \alpha_m} \left[C_m(0) + \sum_{k=1}^{N-m} C_{m+k}(0) \prod_{l=1}^{k} \frac{1}{z - \alpha_{m+l}} \gamma_{m+l-1} \right]. \tag{10.29}$$

Consider next (10.26). Its Laplace transformation yields

$$\tilde{C}_m = \frac{1}{z - \alpha_m} \left[C_m(0) + \beta_m \tilde{C}_{m-1} \right]. \tag{10.30}$$

Set $m = 0$ in this and use $C_{-1} = 0$ to show that

$$\tilde{C}_0 = \frac{1}{z - \alpha_0} C_0(0). \tag{10.31}$$

Solve (10.30) recursively for $m = 1, 2, \ldots$ to obtain

$$\tilde{C}_m = \frac{1}{z - \alpha_m} \left[C_m(0) + \sum_{k=0}^{m-1} C_k(0) \prod_{l=k}^{m-1} \frac{1}{z - \alpha_l} \beta_{l+1} \right]. \tag{10.32}$$

The inverse Laplace transform (defined in (8.85)) of (10.29) and (10.32) determines $C_m(t)$. Now, if α_m are all distinct, then the poles of (10.29) and

(10.32) are simple. The inverse Laplace transform of (10.29) and (10.32) is then straightforward to evaluate. However, it may become an involved exercise if the poles are not simple.

As an example, let $\alpha_m = -(m+\alpha)$. The solution of (10.25) and (10.26) then reads, respectively,

$$C_m(t) = \exp\{-(m+\alpha)t)\}\Big[C_m(0) + \sum_{k=m+1} \frac{(1-\exp(-t))^{k-m}\gamma_m\gamma_{m+1}\cdots\gamma_{k-1}}{(k-m)!}C_k(0)\Big], \tag{10.33}$$

$$C_m(t) = \exp\{-(m+\alpha)t)\}\Big[C_m(0) + \sum_{k=0}^{m-1} \frac{(\exp(t)-1)^{m-k}\beta_{k+1}\beta_{k+2}\cdots\beta_m}{(m-k)!}C_k(0)\Big]. \tag{10.34}$$

The correctness of these solution may be verified by substituting them in their respective equation. We use these solutions in Chap. 14 in solving the problem of strongly coupled atom-cavity system.

10.4 Exactly Solvable Two- and Three-Term Recursion Relations

In this section we solve the eigenvalue equations corresponding to (10.25) and (10.26). It is followed by the solution of the eigenvalue problem reducible to the three-term recursion relation (10.41).

10.4.1 Two-Term Recursion Relations

Consider the problem of determining C_m obeying the eigenvalue equation

$$\alpha_m C_m + \gamma_m C_{m+1} = \lambda C_m, \qquad m = 0, 1, \ldots, \tag{10.35}$$

α_m, γ_m being known functions of m, and λ is an eigenvalue. Rewrite this as

$$(\hat{\alpha} - \lambda I)|C\rangle \equiv \begin{pmatrix} \alpha_0-\lambda & \gamma_0 & 0 & \cdot & \cdot & \cdot \\ 0 & \alpha_1-\lambda & \gamma_1 & 0 & \cdot & \cdot \\ 0 & 0 & \alpha_2-\lambda & \gamma_2 & 0 & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ C_2 \\ \vdots \end{pmatrix} = 0. \tag{10.36}$$

By carrying its expansion in terms of the first column, we see that $\det(\hat{\alpha}-\lambda I)$ in this case is a product of the diagonal elements of $\hat{\alpha} - \lambda I$. Hence, $\det(\hat{\alpha} - \lambda I) = 0$ yields the expression

$$\lambda_n = \alpha_n, \qquad n = 0, 1, \ldots. \tag{10.37}$$

for the eigenvalues. Let C_{nm} denote the solution of (10.35) corresponding to the eigenvalue λ_n. Solve (10.35) recursively to obtain

$$C_{nm} = \prod_{i=1}^{m} \frac{\alpha_n - \alpha_{i-1}}{\gamma_{i-1}} C_{n0}, \qquad m = 1, 2, \cdots, n$$
$$C_{nm+n} = 0 \qquad m = 1, 2, \cdots \tag{10.38}$$

This is the exact solution of (10.35).

Consider next the equation

$$\alpha_m C_m + \beta_m C_{m-1} = \lambda C_m, \qquad m = 0, 1, \ldots \tag{10.39}$$

Invoking the preceeding arguments, it follows that the eigenvalues in this case also are given by (10.37). Verify also that (10.39) is solved by

$$C_{nm+n} = \prod_{i=1}^{m} \frac{\beta_{n+i}}{\alpha_n - \alpha_{n+i}} C_{nn}, \qquad m = 1, 2, \cdots,$$
$$C_{nm} = 0 \qquad m = 0, 1, \cdots, n-1. \tag{10.40}$$

10.4.2 Three-Term Recursion Relations

We have seen in the last subsection that two-term recursion relations admit analytical solutions in closed form. However, three-term relations are not always exactly solvable. In this subsection we identify exactly solvable cases of frequently encountered quadratic three-term recursion relations of the type

$$[m\alpha_1 + m(m-1)\alpha_2]C_m + [(m+1)\gamma_1 + m(m+1)\gamma_2]C_{m+1}$$
$$+[\beta_0 + (m-1)\beta_1 + (m-1)(m-2)\beta_2]C_{m-1} = \lambda C_m, \tag{10.41}$$

$m = 0, 1, \ldots, M$, $C_{-1} = 0$, $\alpha's$, $\beta's$ and $\gamma's$ are fixed constants and λ is an eigenvalue. The upper limit M on the allowed values of m is finite if it is given that $C_{M+k} = 0$ for $k \geq 1$. In case this holds, set $m = M+1$ in (10.41). It results in a relation between C_{M+1}, C_{M+2} and C_M. This equation will be consistent with the given condition $C_{M+1} = C_{M+2} = 0$ if the coefficient of C_M in it vanishes, i.e. if

$$\beta_0 + M\beta_1 + M(M-1)\beta_2 = 0. \tag{10.42}$$

Write this as a quadratic in M and show that

$$M = \frac{1}{2\beta_2}\left[\beta_2 - \beta_1 \pm \sqrt{(\beta_2 - \beta_1)^2 - 4\beta_2\beta_0}\right]. \tag{10.43}$$

We solve (10.41) by converting it into a differential equation for the *generating function*

$$f(x) = \sum_{m=0}^{M} C_m x^m \tag{10.44}$$

so that

$$C_m = \frac{1}{m!}\frac{\mathrm{d}^m}{\mathrm{d}x^m} f(x)\Big|_{x=0}. \tag{10.45}$$

It is then straightforward to show that, by virtue of (10.41), (together with (10.42) if M is finite) $f(x)$ satisfies the second-order differential equation

$$\Big[x(\beta_2 x^2 + \alpha_2 x + \gamma_2)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (\beta_1 x^2 + \alpha_1 x + \gamma_1)\frac{\mathrm{d}}{\mathrm{d}x} + (\beta_0 x - \lambda)\Big] f(x) = 0. \tag{10.46}$$

The solution of this equation determines C_m through the relation (10.45).

Exact power series solution of a second-order differential equation, reducible to a hypergeometric or a confluent hypergeometric equation, is known. We recall from Appendix B that an ordinary second-order differential equation admits solution in terms of the hypergeometric function if it has at most three singularities, including a singularity at $x = \infty$, which are regular and that its solution is expressible in terms of the confluent hypergeometric function if two of the three singularities merge. The problem of solving (10.46) is thus reduced to one of determining the nature of its singularities. Recall from the Appendix B that the nature of the singularity at $x = \infty$ is determined by transforming to $y = 1/x$. Use (B.2) to show that the change of variable $x = 1/y$ transforms (10.46) to

$$\Big[y(\gamma_2 y^2 + \alpha_2 y + \beta_2)\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \{(2\gamma_2 - \gamma_1)y^2 + (2\alpha_2 - \alpha_1)y + (2\beta_2 - \beta_1)\}\frac{\mathrm{d}}{\mathrm{d}y} + \frac{1}{y}(\beta_0 - \lambda y)\Big] f(y) = 0. \tag{10.47}$$

The nature of the singularity of this at $y = 0$ determines that of the point $x = \infty$ of (10.46).

Following again the Appendix B, we find that if $\beta_2 \neq 0$, $\gamma_2 \neq 0$ then

$$x = 0, \qquad x = \frac{-\alpha_2 \pm \sqrt{\alpha_2^2 - 4\beta_2\gamma_2}}{2\beta_2}, \qquad x = \infty \tag{10.48}$$

are four singular points of (10.46). However, as mentioned above, it is the case of at most three regular singularities which is of interest. The number of singularities reduces to three if (A) $\beta_2 = 0$, $\gamma_2 \neq 0$, $\alpha_2 \neq 0$, or if (B) $\gamma_2 = 0$, $\beta_2 \neq 0$, $\alpha_2 \neq 0$, or if (C) $\alpha_2^2 = 4\beta_2\gamma_2 \neq 0$.

The number of singular points is two if (D) $\gamma_2 \neq 0$, $\beta_2 = 0$, $\alpha_2 = 0$ or if (E) $\beta_2 \neq 0$, $\gamma_2 = 0$, $\alpha_2 = 0$ or if (F) $\beta_2 = \gamma_2 = 0$.

Finally, we note that (10.46) reduces to a first-order equation if $\alpha_2 = \beta_2 = \gamma_2 = 0$. We discuss this below as case (G).

Case A: $\beta_2 = 0$, $\gamma_2 \neq 0$, $\alpha_2 \neq 0$. In this case, the singularities of (10.46) are at

$$x = 0, \qquad x = -\frac{\gamma_2}{\alpha_2}, \qquad x = \infty. \tag{10.49}$$

Verify that the first two singularities above are regular whereas the singularity at $x = \infty$ is regular only if $\beta_0 = \beta_1 = 0$. Under these conditions, the three-term recursion relation (10.41) reduces to a two-term relation

$$[m\alpha_1 + m(m-1)\alpha_2]C_m + (m+1)[\gamma_1 + m\gamma_2]C_{m+1} = \lambda C_m. \tag{10.50}$$

This is a special case of already solved equation (10.35).

Case B: $\gamma_2 = 0$, $\beta_2 \neq 0$, $\alpha_2 \neq 0$. The singularities of (10.46) in this case are at

$$x = 0, \qquad x = -\frac{\alpha_2}{\beta_2}, \qquad x = \infty. \tag{10.51}$$

Verify that the last two singularities above are regular whereas the singularity at $x = 0$ is regular only if $\gamma_1 = 0$. Under these conditions, (10.41) reduces to

$$\begin{aligned}&[m\alpha_1 + m(m-1)\alpha_2]C_m \\ &+[\beta_0 + (m-1)\{\beta_1 + (m-2)\beta_2\}]C_{m-1} = \lambda C_m\end{aligned} \tag{10.52}$$

This is a special case of already solved equation (10.39).

Case C: $\alpha_2^2 = 4\beta_2\gamma_2 \neq 0$. , The singularities of (10.46) in this case are at

$$x = 0, \; x = -\frac{\alpha_2}{2\beta_2}, \; x = \infty. \tag{10.53}$$

Verify that the first and the last singularity above are regular whereas the second singularity is regular if the coefficient of $\mathrm{d}f/\mathrm{d}x$ can be factorized as

$$\beta_1 x^2 + \alpha_1 x + \gamma_1 = \beta_1 \left(x + \frac{\alpha_2}{2\beta_2}\right)(x + K), \tag{10.54}$$

i.e. if

$$\beta_1\Big(K + \frac{\alpha_2}{2\beta_2}\Big) = \alpha_1, \qquad K = \frac{2\gamma_1\beta_2}{\beta_1\alpha_2}. \tag{10.55}$$

The two equations above determining one unknown K are consistent if

$$\frac{\alpha_1}{\beta_1} - \frac{\alpha_2}{2\beta_2} = \frac{2\gamma_1\beta_2}{\alpha_2\beta_1}. \tag{10.56}$$

The differential equation (10.46) then reduces to

$$\begin{aligned}&\Big[x\beta_2\Big(x + \frac{\alpha_2}{2\beta_2}\Big)^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \beta_1\Big(x + \frac{\alpha_2}{2\beta_2}\Big)\Big(x + K\Big)\frac{\mathrm{d}}{\mathrm{d}x} \\ &+\Big(\beta_0 x - \lambda\Big)\Big]f(x) = 0.\end{aligned} \tag{10.57}$$

Verify that the transformation $x = -\alpha_2 z/2\beta_2$ reduces this equation to the form (B.7) with $p = -\beta_1/\beta_2$, $q = -2K\beta_2/\alpha_2$, $r = \beta_0/\beta_2$, $\lambda \to -2\lambda/\alpha_2$. A solution of (10.57) is, therefore, given by

$$f(x) = (1 + 2\beta_2 x/\alpha_2)^{\alpha}\, F\left(a, b; c; -2\beta_2 x/\alpha_2\right), \tag{10.58}$$

with $c = 2K\beta_1/\alpha_2$, and a, b and α determined by solving

$$\alpha(\alpha-1)+\frac{\beta_1}{\beta_2}\alpha+\frac{\beta_0}{\beta_2}=\frac{2}{\alpha_2}(K\beta_1\alpha-\lambda), \tag{10.59}$$

$$a+b+1=2\alpha+\frac{\beta_1}{\beta_2}, \qquad ab=\frac{2}{\alpha_2}(K\beta_1\alpha-\lambda). \tag{10.60}$$

Let us assume that $C_{M+k}=0$ $(k\geq 1)$ so that (10.42) holds. The $f(x)$ in this case should be a polynomial of degree M. Recall from Appendix B that $F(a,b;c;x)$ is a polynomial of degree n if either a or b is $-n$. Hence, we must have

$$a=-n, \qquad \alpha=M-n. \tag{10.61}$$

On combining this with (10.60) we find that

$$b=2M-n+\frac{\beta_1}{\beta_2}-1,$$

$$\lambda=\frac{n\alpha_2}{2}\left[2M-n+\frac{\beta_1}{\beta_2}-1\right]+K\beta_1(M-n). \tag{10.62}$$

The expression for C_m corresponding to the eigenvalue λ_n is obtained by substituting (10.58) in (10.45).

Case D: $\gamma_2\neq 0$, $\alpha_2=\beta_2=0$. The singularities of (10.46) in this case are at $x=0$, $x=\infty$. The singularity at $x=0$ is regular whereas that at $x=\infty$ is irregular. The equation (10.46) is reducible to the equation (B.13) for the confluent hypergeometric function if, in addition, $\beta_0=\beta_1=0$. Note that this is obtained as the limit $\alpha_2\to 0$ of the case A above. In this limit, the singularity at $x=-\gamma_2/\alpha_2$ of case A merges with already present singularity at ∞. The recursion relation (10.41) now reads

$$\alpha_1 mC_m+(m+1)(\gamma_1+m\gamma_2)C_{m+1}=\lambda C_m. \tag{10.63}$$

This is a special case of already solved equation (10.35).

Case E: $\beta_2\neq 0$, $\alpha_2=\gamma_2=0$. The singularities in this case are at $x=0$ and $x=\infty$. The singularity at $x=\infty$ is regular whereas that at $x=0$ is always irregular. Since we are interested only in the case of a regular singularity at $x=0$, this case is not of interest.

Case F: $\beta_2=\gamma_2=0$. The singularities in this case are at $x=0$, $x=\infty$. Both the singularities are irregular. However, the singularity at $x=0$ becomes regular if $\gamma_1=0$. The corresponding recursion relation assumes the form

$$[m\alpha_1+m(m-1)\alpha_2-\lambda]C_m+[\beta_0+(m-1)\beta_1]C_{m-1}=0. \tag{10.64}$$

This is a special case of already solved equation (10.39).

Case (G) $\alpha_2 = \beta_2 = \gamma_2 = 0$. The equation (10.46) in this case reduces to a first order equation

$$\left[(\beta_1 x^2 + \alpha_1 x + \gamma_1)\frac{\mathrm{d}}{\mathrm{d}x} + (\beta_0 x - \lambda)\right] f(x) = 0. \tag{10.65}$$

The recursion relation (10.41) then assumes the form

$$m\alpha_1 C_m + (m+1)\gamma_1 C_{m+1} + [\beta_0 + (m-1)\beta_1]C_{m-1} = \lambda C_m. \tag{10.66}$$

Verify that the solution of (10.65) is

$$f(x) \sim (x - x_+)^a (x - x_-)^b, \tag{10.67}$$

$$x_\pm = \frac{1}{2\beta_1}\left[-\alpha_1 \pm \sqrt{\alpha_1^2 - 4\gamma_1\beta_1}\right],$$
$$a = \frac{\lambda - \beta_0 x_+}{\beta_1(x_+ - x_-)}, \qquad b = \frac{-\lambda + \beta_0 x_-}{\beta_1(x_+ - x_-)}. \tag{10.68}$$

Let $C_{M+n} = 0$ $(n \geq 1)$. The $f(x)$ in (10.67) then should be a polynomial of degree M. This implies that a and b should be integers such that $a + b = M$. We set $a = n$, $b = M - n$. Now apply (10.45) to get

$$C_{mn} \sim (-)^m \sum_k \frac{(x_+)^{n-k}(x_-)^{M-n-m+k}}{(m-k)!k!(n-k)!(M-n-m+k)!}. \tag{10.69}$$

We have used this result in Chap. 3 for deriving the expression for the squeezed spin state.

11. Two-Level and Three-Level Hamiltonian Systems

In this chapter we present exact analytical approach to studying the dynamical behaviour of certain classes of two-level and three-level hamiltonian systems in a quantized e.m. field. These classes encompass most of commonly encountered systems of interest in quantum optics. This chapter is based largely on [101].

11.1 Exactly Solvable Two-Level Systems

A class of exactly solvable system of a two-level atom in a quantized field is comprised by the hamiltonian

$$\hat{H}_1 = \hbar\Big[\hat{h}_0(\{\hat{C}\},t) + \delta(\{\hat{C}\},t)\hat{S}_z + \Big\{g(\{\hat{C}\},t)\hat{F}^\dagger\hat{S}_- + \text{h.c.}\Big\}\Big] \tag{11.1}$$

in an appropriate interaction picture. Here, $\hat{F}, \hat{F}^\dagger$ are time-independent sums of products of single mode field operators whereas $\{\hat{C}\}$ is a set of commuting operators each commuting with $\hat{S}_z$, $\hat{F}^\dagger\hat{S}_-$ and $\hat{S}_+\hat{F}$. In other words, $\{\hat{C}\}$ is a set of commuting time-independent constants. In the following we suppress displaying explicit dependence on $\{\hat{C}\}$. The Hamiltonian (11.1) describes emission and absorption of the field quanta by the effective field operators $\hat{F}$ and $\hat{F}^\dagger$. The effective detuning $\delta(t)$ and the atom-field coupling constant $g(t)$ may be time-dependent. The properties of two-level operators are contained in the equations (1.129)–(1.134b).

The time-evolution operator generated by $\hat{H}_1$ is

$$\hat{U}_1(t) = \overleftarrow{T}\exp\Big[-\frac{\mathrm{i}}{\hbar}\int_0^t \hat{H}_1(\tau)\mathrm{d}\tau\Big]. \tag{11.2}$$

We follow the method of Chap. 2 and write the time-ordered exponential (11.2) in terms of the products of the exponentials of all those operators which, along with the operators $\hat{S}_z$, $\hat{F}^\dagger\hat{S}_-$ and $\hat{S}_+\hat{F}$ constituting $\hat{H}_1(t)$, are closed under the operation of commutation. To that end, recall the properties (1.129)–(1.134b) of two-level operators and show that

$$[\hat{S}_z, \hat{F}^\dagger\hat{S}_-] = -\hat{F}^\dagger\hat{S}_-, \qquad [\hat{S}_z, \hat{S}_+\hat{F}] = \hat{S}_+\hat{F}, \tag{11.3}$$

$$[\hat{S}_+\hat{F}, \hat{F}^\dagger\hat{S}_-] = \hat{S}_+\hat{S}_-\hat{F}\hat{F}^\dagger - \hat{F}^\dagger\hat{F}\hat{S}_-\hat{S}_+. \tag{11.4}$$

Now, recall (1.134b) to rewrite $\hat{S}_+\hat{S}_-$ and $\hat{S}_-\hat{S}_+$ respectively as $2\hat{S}_z\hat{S}_+\hat{S}_-$ and $-2\hat{S}_z\hat{S}_-\hat{S}_+$. Consequently, (11.4) may be rewritten as

$$[\hat{S}_+\hat{F}, \hat{F}^\dagger\hat{S}_-] = 2\hat{S}_z\hat{N}, \tag{11.5}$$

$$\hat{N} = \hat{F}\hat{F}^\dagger\hat{S}_+\hat{S}_- + \hat{F}^\dagger\hat{F}\hat{S}_-\hat{S}_+. \tag{11.6}$$

It is straightforward to see that

$$\hat{N}\hat{S}_- = \hat{F}^\dagger\hat{F}\hat{S}_-, \qquad \hat{N}\hat{S}_+ = \hat{F}\hat{F}^\dagger\hat{S}_+. \tag{11.7}$$

By repeated use of these relations, it follows that

$$\Phi(\hat{N})\hat{S}_- = \Phi(\hat{F}^\dagger\hat{F})\hat{S}_-, \qquad \Phi(\hat{N})\hat{S}_+ = \Phi(\hat{F}\hat{F}^\dagger)\hat{S}_+, \tag{11.8}$$

for any $\Phi\hat{N}$ expandable as a power series in its argument.

The reader should also verify that $\hat{N}$ commutes with $\hat{S}_+\hat{F}$, $\hat{F}^\dagger\hat{S}_-$ and, of course, with $\hat{S}_z$. Hence $\hat{N}$ commutes with each of the operators in $\hat{H}_1(t)$. It is, therefore, a time-independent constant of the motion. Hence, the time-ordered exponential (11.2) may be written as

$$\begin{aligned}
\hat{U}_1 &= \exp\Big[-\mathrm{i}\int_0^t \hat{h}_0(\tau)\mathrm{d}\tau\Big]\Big[\exp\Big(\hat{f}_+(t)\hat{S}_+\hat{F}\Big)\exp\Big(2f_z(t)\hat{S}_z\Big) \\
&\quad\times\exp\Big(\hat{f}_-(t)\hat{F}^\dagger\hat{S}_-\Big)\Big] \\
&= \exp\Big[-\mathrm{i}\int_0^t \hat{h}_0(\tau)\mathrm{d}\tau\Big]\Big(1+\hat{f}_+(t)\hat{S}_+\hat{F}\Big) \\
&\quad\times\Big(\cosh(\hat{f}_z(t))+2\sinh(\hat{f}_z(t))\hat{S}_z\Big)\Big(1+\hat{f}_-(t)\hat{F}^\dagger\hat{S}_-\Big) \\
&= \exp\Big[-\mathrm{i}\int_0^t \hat{h}_0(\tau)\mathrm{d}\tau\Big]\Big[\hat{\alpha}_0(t)+\hat{x}_{++}(t)\hat{S}_+\hat{F}+\hat{x}_{--}(t)\hat{F}^\dagger\hat{S}_- \\
&\quad+\hat{\alpha}_z(t)\hat{S}_z+\hat{\alpha}_{+-}(t)\hat{S}_+\hat{S}_-\hat{F}\hat{F}^\dagger\Big].
\end{aligned} \tag{11.9}$$

Here, $\hat{f}_\pm(t)$ and $\hat{f}_z(t)$ are time-dependent functions of time-independent operator constants of the motion. The second equation above has been obtained by applying (2.13) and the third by invoking the two-level operator relations. The α's and the x's may be identified in terms of the f's. However, we do not need that relation. We write $\hat{S}_z$ as $(1/2)[\hat{S}_+, \hat{S}_-]$, $\hat{\alpha}_0(t)$ as $\hat{\alpha}_0(t)(\hat{S}_+\hat{S}_- + \hat{S}_-\hat{S}_+)$, and invoke (11.7) to rewrite (11.9) as

$$\begin{aligned}
\hat{U}_1 &= \exp\Big[-\mathrm{i}\int_0^t \hat{h}_0(\tau)\mathrm{d}\tau\Big]\Big[\hat{x}_{+-}(t)\hat{S}_+\hat{S}_- + \hat{x}_{-+}(t)\hat{S}_-\hat{S}_+ \\
&\quad+\hat{x}_{++}(t)\hat{S}_+\hat{F}+\hat{x}_{--}(t)\hat{F}^\dagger\hat{S}_-\Big],
\end{aligned} \tag{11.10}$$

$\hat{x}_{ij}(t)$ being the functions of the time-independent operator constants of the motion. Note that $\hat{U}_1$ for a classically driven two-level system has the same form as (11.10) if the quantum field operator $\hat{F}$ is replaced by a c-number.

The time-dependence of the evolution is contained in the $\hat{x}_{ij}(t)$ which are determined next.

To derive $\hat{x}_{ij}(t)$, write $\hat{U}_1$ in (11.10) as in (11.2), differentiate it with respect to t and use (11.1) for $\hat{H}_1(t)$ to get

$$\begin{aligned}&\left[\delta(t)\hat{S}_z + g(t)\hat{F}^\dagger\hat{S}_- + g^*(t)\hat{S}_+\hat{F}\right]\hat{U}_1\\ &\quad = \mathrm{i}\exp\left[-\mathrm{i}\int_0^t \hat{h}_0(\tau)\mathrm{d}\tau\right]\left[\dot{\hat{x}}_{+-}(t)\hat{S}_+\hat{S}_- + \dot{\hat{x}}_{++}(t)\hat{S}_+\hat{F}\right.\\ &\qquad \left.+\dot{\hat{x}}_{--}(t)\hat{F}^\dagger\hat{S}_- + \dot{\hat{x}}_{-+}(t)\hat{S}_-\hat{S}_+\right],\end{aligned} \tag{11.11}$$

where 'dot' over a quantity denotes derivative with respect to time. On substituting for $\hat{U}_1$ from (11.10), (11.11) reads

$$\begin{aligned}&\mathrm{i}\left[\dot{\hat{x}}_{+-}\hat{S}_+\hat{S}_- + \dot{\hat{x}}_{++}\hat{S}_+\hat{F} + \dot{\hat{x}}_{--}\hat{F}^\dagger\hat{S}_- + \dot{\hat{x}}_{-+}\hat{S}_-\hat{S}_+\right]\\ &\quad = \left[\delta(t)\hat{S}_z + g(t)\hat{F}^\dagger\hat{S}_- + g^*(t)\hat{S}_+\hat{F}\right]\\ &\qquad \times\left[\hat{x}_{+-}\hat{S}_+\hat{S}_- + \hat{x}_{++}\hat{S}_+\hat{F} + \hat{x}_{--}\hat{F}^\dagger\hat{S}_- + \hat{x}_{-+}\hat{S}_-\hat{S}_+\right].\end{aligned} \tag{11.12}$$

On applying the two-level characteristics and the relations (11.7), we express the right hand side of (11.12) as a combination of the same operators that appear on its left hand side. On comparing the coefficients of the like operators we arrive at two closed systems of equations

$$\begin{aligned}\mathrm{i}\dot{\hat{x}}_{++} &= \frac{\delta(t)}{2}\hat{x}_{++} + g^*(t)\hat{x}_{-+},\\ \mathrm{i}\dot{\hat{x}}_{-+} &= -\frac{\delta(t)}{2}\hat{x}_{-+} + \hat{N}g(t)\hat{x}_{++};\end{aligned} \tag{11.13}$$

and

$$\begin{aligned}\mathrm{i}\dot{\hat{x}}_{--} &= -\frac{\delta(t)}{2}\hat{x}_{--} + g(t)\hat{x}_{+-},\\ \mathrm{i}\dot{\hat{x}}_{+-} &= \frac{\delta(t)}{2}\hat{x}_{+-} + \hat{N}g^*(t)\hat{x}_{--}.\end{aligned} \tag{11.14}$$

The sets of equations (11.13) and (11.14) are similar. They are to be solved along with the initial condition

$$\hat{x}_{++}(0) = \hat{x}_{--}(0) = 0, \qquad \hat{x}_{+-}(0) = \hat{x}_{-+}(0) = 1. \tag{11.15}$$

Note that, since $\hat{N}$ is a constant operator, (11.13) and (11.14) may be treated like c-number equations. Their solution for given time-dependent functions $g(t)$ and $\delta(t)$ determines $\hat{x}_{ij}$ for any field operator $\hat{F}$ in the hamiltonian (11.1).

Dynamics of the system is thus determined by solving the c-number like equations (11.13) and (11.14) for $\hat{x}_{ij}$. The $\hat{x}_{ij}$ are functions of $\hat{N}$ which contains, besides the field operators, also the atomic operators. Since $\hat{x}_{ij}$ appear in (11.10) only in combination with $\hat{S}_\pm$, we can get rid of the dependence

of $\hat{x}_{ij}$ on the atomic operators by using (11.8). The $\hat{x}_{ij}$ may depend also on other constants of motion which may, in turn, contain the atomic operators. Such atomic operators can also be removed by using the relations (11.8) and, possibly, other similar relations.

Assuming that the other constants, if any, do not involve field operators in any other combination then what appears in $\hat{N}$, the $\hat{x}_{ij}$ turn out to be functions of the field operators $\hat{F}^\dagger\hat{F}$ and $\hat{F}\hat{F}^\dagger$. The problem of evaluating matrix elements of $\hat{U}_1$ reduces to that of solving the eigenvalue problem of $\hat{F}^\dagger\hat{F}$ and $\hat{F}\hat{F}^\dagger$. These two eigenvalue problems are, however, not independent. For, if $|\psi\rangle$ is an eigenstate of $\hat{F}^\dagger\hat{F}$ then it follows that $\hat{F}|\psi\rangle$ is an eigenstate of $\hat{F}\hat{F}^\dagger$ corresponding to the same eigenvalue. Note also that $\hat{F}^\dagger\hat{F}$ and $\hat{F}\hat{F}^\dagger$ are normal operators. Hence, their eigenstates constitute a complete set which may be used as the basis states. The eigenvalue problem in question can be solved effortlessly for single-channel models. For, recall from Sect. 7.5 that $\hat{F}$ then is a product of single-mode operators and hence $\hat{F}^\dagger\hat{F}$ a product of number operators. The eigenvalue problem, however, may not be simple for multi-channel models (see (11.47)).

Now, the time-evolution of a state of the system is determined by

$$|\psi(t)\rangle = \hat{U}_1(t)|\psi(0)\rangle. \tag{11.16}$$

If the system is described by a density operator $\hat{\rho}$ then

$$\hat{\rho}(t) = \hat{U}_1(t)\hat{\rho}(0)\hat{U}_1^\dagger(t), \tag{11.17}$$

$\hat{\rho}(0)$ being the density operator at the initial time. The dynamics may be investigated in terms of evolution of operators by evaluating

$$\hat{A}(t) = \hat{U}_1^\dagger(t)\hat{A}\hat{U}_1(t), \tag{11.18}$$

$\hat{A}$ being a system operator.

Next, we examine the equations (11.13) and (11.14). Those equations can be solved analytically exactly if (a) δ and g are time-independent, or if (b) $\delta = 0$ and $g(t)$ is real. In the following we derive the expression for $\hat{U}_1(t)$ for these exactly solvable cases, and for the case of (c) random time-dependence of $\delta(t)$ and $g(t)$.

11.1.1 Time-Independent Detuning and Coupling

The solution of equations (11.13) and (11.14) in this case is readily obtained reading

$$\begin{aligned}
\hat{x}_{+-} &= \cos(\hat{\Gamma}t) - \mathrm{i}\frac{\delta}{2}\hat{\Gamma}^{-1}\sin(\hat{\Gamma}t),\\
\hat{x}_{-+} &= \cos(\hat{\Gamma}t) + \mathrm{i}\frac{\delta}{2}\hat{\Gamma}^{-1}\sin(\hat{\Gamma}t),\\
\hat{x}_{++} &= -\mathrm{i}g^*\hat{\Gamma}^{-1}\sin(\hat{\Gamma}t), \qquad \hat{x}_{--} = -\mathrm{i}g\hat{\Gamma}^{-1}\sin(\hat{\Gamma}t),
\end{aligned} \tag{11.19}$$

$$\hat{\Gamma}^2 = \frac{1}{4}\Big[\delta^2 + 4|g|^2\hat{N}\Big]. \tag{11.20}$$

On substituting (11.19) in (11.10) and on applying (11.8) we obtain

$$\begin{aligned}\hat{U}_1(t) = \exp(-\mathrm{i}\hat{h}_0 t)\Big\{&\Big[\cos(\hat{\Gamma}_2 t) - \mathrm{i}\frac{\delta}{2}\hat{\Gamma}_2^{-1}\sin(\hat{\Gamma}_2 t)\Big]\hat{S}_+\hat{S}_- \\ &+\Big[\cos(\hat{\Gamma}_1 t) + \mathrm{i}\frac{\delta}{2}\hat{\Gamma}_1^{-1}\sin(\hat{\Gamma}_1 t)\Big]\hat{S}_-\hat{S}_+ \\ &-\mathrm{i}\Big[g\hat{\Gamma}_1^{-1}\sin(\hat{\Gamma}_1 t)\hat{F}^\dagger\hat{S}_- + g^*\hat{\Gamma}_2^{-1}\sin(\hat{\Gamma}_2 t)\hat{F}\hat{S}_+\Big]\Big\},\end{aligned} \tag{11.21}$$

$$\hat{\Gamma}_1^2 = \frac{1}{4}\Big[\delta^2 + 4|g|^2\hat{F}^\dagger\hat{F}\Big], \qquad \hat{\Gamma}_2^2 = \frac{1}{4}[\delta^2 + 4|g|^2\hat{F}\hat{F}^\dagger\Big]. \tag{11.22}$$

This determines completely the dynamics generated by time-independent form of the hamiltonian (11.1). It is also of interest to know its eigenstates called its dressed states.

Dressed States. Let $|\lambda\rangle$ be the eigenstates of $\hat{F}\hat{F}^\dagger$ corresponding to the eigenvalue λ:

$$\hat{F}\hat{F}^\dagger|\lambda\rangle = \lambda|\lambda\rangle. \tag{11.23}$$

Since $\hat{H}_1$ commutes with $\hat{N}$, we may reduce the space of states to a sum of subspaces each characterized by an eigenvalue of $\hat{N}$. To that end, verify that

$$\hat{N}|\lambda, 1/2\rangle = \lambda|\lambda, 1/2\rangle, \qquad \hat{N}\hat{F}^\dagger|\lambda, -1/2\rangle = \lambda\hat{F}^\dagger|\lambda, -1/2\rangle. \tag{11.24}$$

Hence, the states $|\lambda, 1/2\rangle$ and $\hat{F}^\dagger|\lambda, -1/2\rangle$ correspond to the same eigenvalue λ of $\hat{N}$. The space of the eigenstates of $\hat{H}_1$ is thus split in to manifolds characterized by an eigenvalue of $\hat{N}$. Each manifold consists of two states, $|\lambda, 1/2\rangle$ and $\hat{F}^\dagger|\lambda, -1/2\rangle$.

We may now express the eigenstates of time-independent form of (11.1) as

$$|\mu\rangle = \cos(\theta)|\lambda, 1/2\rangle + \frac{\sin(\theta)}{\sqrt{\lambda}}\exp(\mathrm{i}\phi)\hat{F}^\dagger|\lambda, -1/2\rangle, \tag{11.25}$$

with $\exp(\mathrm{i}\phi) = g/|g|$. Let

$$\hat{H}_1|\mu\rangle = \hbar\mu|\mu\rangle. \tag{11.26}$$

Redefine the eigenvalues of $\hat{H}_1$ by absorbing in them the constant contribution from $\hat{h}_0$ to obtain

$$\begin{aligned}\hat{H}_1|\mu\rangle = \hbar\Bigg[&\Big(\cos(\theta)\frac{\delta}{2} + |g|\sqrt{\lambda}\sin(\theta)\Big)|\lambda, 1/2\rangle \\ &+\frac{1}{\sqrt{\lambda}}\Big(|g|\sqrt{\lambda}\cos(\theta) - \sin(\theta)\frac{\delta}{2}\Big)\exp(\mathrm{i}\phi)\hat{F}^\dagger|\lambda, -1/2\rangle\Bigg].\end{aligned} \tag{11.27}$$

Substitute this in (11.26) to arrive at the eigenvalue equation

$$\begin{pmatrix} \mu - \delta/2 & -|g|\sqrt{\lambda} \\ -|g|\sqrt{\lambda} & \mu + \delta/2 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} = 0. \tag{11.28}$$

It is now straightforward to see that the eigenvalues are

$$\mu_{\pm} = \pm\sqrt{\lambda|g|^2 + \delta^2/4}, \tag{11.29}$$

and that

$$\tan(\theta) = \frac{\mu_{\pm} - \delta/2}{|g|\sqrt{\lambda}}. \tag{11.30}$$

The states $|\mu_{\pm}\rangle$ are known as the *dressed states* of the hamiltonian.

If $\delta = 0$ then $\tan(\theta) = \pm 1$. Hence (11.25) reduces to

$$|\mu_{\pm}\rangle = \frac{1}{\sqrt{2}}\left[|\lambda, 1/2\rangle \pm \frac{1}{\sqrt{\lambda}} \exp(\mathrm{i}\phi)\hat{F}^{\dagger}|\lambda, -1/2\rangle\right]. \tag{11.31}$$

Now we specialize these results to (a) the Jaynes–Cummings model of one-photon transitions, and (c) a two-channel Raman coupled model.

Jaynes–Cummings Model. The hamiltonian (11.1) reduces to the Jaynes–Cummings hamiltonian (7.42) of a two-level atom in a single mode field in the interaction picture generated by $\hbar\omega\hat{S}_z$ with $\delta = \omega_0 - \omega$, $\hat{F} \to \hat{a}$. Note also that in this case $\hat{N} = \hat{a}^{\dagger}\hat{a} + \hat{S}_z + 1/2$. The eigenstates and eigenvalues of $\hat{a}^{\dagger}\hat{a}$ are $|n\rangle$ and $n = 0, 1, \ldots$. The dressed states of this model for an arbitrary δ may be derived by using (11.25) and bearing in mind that $|\lambda\rangle = |m\rangle$ where $|m\rangle$ is an eigenstate of $\hat{a}\hat{a}^{\dagger}$ with $m + 1$ as the corresponding eigenvalue. The dressed states for $\delta = 0$, given by (11.31), now read (assuming g to be real)

$$|\psi_m^{(\pm)}\rangle = \frac{1}{\sqrt{2}}\left[|m, 1/2\rangle \pm |m + 1, -1/2\rangle\right]. \tag{11.32}$$

The eigenvalues corresponding to $|\psi_m^{(\pm)}\rangle$ are

$$\mu_m^{(\pm)} = \pm|g|\sqrt{m + 1}. \tag{11.33}$$

The inverse of the relations (11.32) is

$$\begin{aligned} |m, 1/2\rangle &= \frac{1}{\sqrt{2}}\left[|\psi_m^{(+)}\rangle + |\psi_m^{(-)}\rangle\right], \\ |m + 1, -1/2\rangle &= \frac{1}{\sqrt{2}}\left[|\psi_m^{(+)}\rangle - |\psi_m^{(-)}\rangle\right]. \end{aligned} \tag{11.34}$$

We may now investigate the dynamics by expressing an initial state in terms of the dressed states. As an example, let the system be initially in the state $|m, 1/2\rangle$. Using (11.34) we get

$$\begin{aligned} \exp(-\mathrm{i}\hat{H}_1 t/\hbar)|m, 1/2\rangle = \frac{1}{\sqrt{2}}\Big[&\exp(-\mathrm{i}|g|t\sqrt{m + 1})|\psi_m^{(+)}\rangle \\ &+ \exp(\mathrm{i}|g|t\sqrt{m + 1})|\psi_m^{(-)}\rangle\Big]. \end{aligned} \tag{11.35}$$

Express $|\psi_m^{(\pm)}\rangle$ in terms of the bare states $|m, \pm 1/2\rangle$ using (11.32) to rewrite (11.35) as

$$\begin{aligned}\exp(-\mathrm{i}\hat{H}_1 t/\hbar)|m, 1/2\rangle = &\Big[\cos(|g|t\sqrt{m+1}))|m, 1/2\rangle \\ &-\mathrm{i}\sin(|g|t\sqrt{m+1})|m+1, -1/2\rangle\Big] \\ &\equiv |m, 1/2, t\rangle. \end{aligned} \tag{11.36}$$

Now, if the atom is initially in its excited state and the field in a state described by $\hat{\rho}_f(0)$ then

$$\hat{\rho}(0) = \sum_{m,n} C_{mn}|m, 1/2\rangle\langle n, 1/2|, \quad \hat{\rho}_f(0) = \sum_{m,n} C_{mn}|m\rangle\langle n|. \tag{11.37}$$

On applying (11.36) it follows that

$$\hat{\rho}(t) = \sum_{m,n} C_{mn}|m, 1/2, t\rangle\langle n, 1/2, t| \quad \hat{\rho}_f(0) = \sum_{m,n} C_{mn}|m\rangle\langle n|. \tag{11.38}$$

It is now straightforward to evaluate operator averages. For example,

$$\langle \hat{S}_z(t)\rangle = \frac{1}{2}\sum_{m=0}^{\infty}\langle m|\hat{\rho}_f(0)|m\rangle\cos(2gt\sqrt{m+1}). \tag{11.39}$$

The corresponding result when the atom is initially in its ground state reads

$$\langle \hat{S}_z(t)\rangle = -\frac{1}{2}\sum_{m=0}^{\infty}\langle m|\hat{\rho}_f(0)|m\rangle\cos(2gt\sqrt{m}). \tag{11.40}$$

For a discussion of comparison between the behaviour of $\langle \hat{S}_z(t)\rangle$ depicted by (11.39) and (11.40) in a quantized field with that when the field is a classical dynamic variable, see [80]. Here we compare the evolution in the quantized field with that in an externally prescribed monochromatic field. The hamiltonian of the system in the RWA is then given by (7.24). The atomic operators under it evolve as in (7.29) and (7.30). We let $\delta = 0$ and find that, if the atom is initially in the excited or in the ground state, then

$$\langle \hat{S}_z(t)\rangle = = \pm\frac{1}{2}\cos(2|g||\alpha|t). \tag{11.41}$$

In the following we compare the characteristics of the evolution described by (11.39)–(11.41).

1. If the field is initially in the Fock state $|M\rangle$ then $\langle m|\hat{\rho}_f(0)|m\rangle = \delta_{mM}$. Hence $\langle \hat{S}_z(t)\rangle$ oscillates sinusoidally. This behaviour is the same as that of the atom in an external classical field described by (11.41).
2. The expression (11.39) for the atom initially in the excited state shows that $\langle \hat{S}_z(t)\rangle$ exhibits oscillations even when the field is initially in the vacuum state $|0\rangle$. These are called the *vacuum field Rabi oscillations.* However, the expression (11.40) for the atom initially in the ground state

shows that it remains in the ground state if the field is initially in the state of the vacuum. The expression (11.41) for the atom in an external field shows that it remains in its initial state in the absence of an applied field. The vacuum field Rabi oscillations are thus a signature of the field quantization.

3. From (11.39) we infer that the time evolution of $\langle \hat{S}_z(t) \rangle$ in a field which is not in a Fock state is a result of combination of frequencies proportional to $\sqrt{m+1}$. As examples, we consider the coherent and the thermal states of the field. Recall that

$$\langle m|\hat{\rho}_f(0)|m\rangle = \exp(-|\alpha|^2)|\alpha|^{2m}/m! \tag{11.42}$$

if the field is in the coherent state $|\alpha\rangle$ and

$$\langle m|\hat{\rho}_f(0)|m\rangle = \bar{n}^m/(\bar{n}+1)^{m+1} \tag{11.43}$$

if the field is in the thermal state with $\bar{n}$ as the mean number of photons in it. The sum in (11.39), (11.40) can not be carried analytically exactly for either of these states. Numerical results for the field in the coherent state and the atom initially in the lower state are presented in the Fig. 11.1. It exhibits the phenomenon of *collapses and revivals*. This phenomenon has been analyzed analytically under the condition $|\alpha| \gg 1$ in [102, 103]. The behaviour of $\langle \hat{S}_z(t) \rangle$ as a function of time in the thermal field and the atom initially in its lower state is exhibited in Fig. 11.2 by evaluating (11.40) read with (11.43). It is seen that $\langle \hat{S}_z(t) \rangle$ rises to become positive and then collapses to oscillate around zero. The nature of oscillations is evidently very much different from that in the coherent state. The analysis of the sum by following the technique employed for the coherent field predicts the collapse but not the revivals [103, 104].

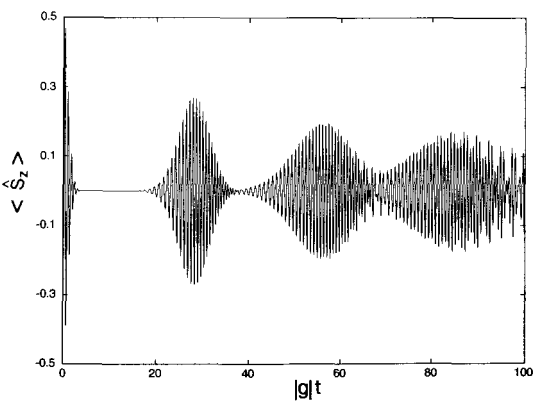

Fig. 11.1. $\langle \hat{S}_z(t) \rangle$ as a function of $|g|t$ for the field in the coherent state $|\alpha\rangle$, $|\alpha|^2 = 20$.

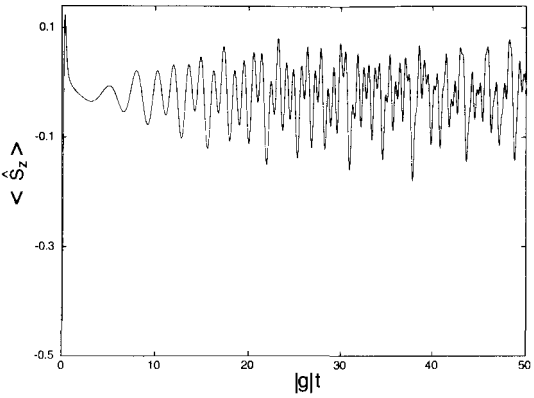

Fig. 11.2. $\langle \hat{S}_z(t) \rangle$ as a function of $|g|t$ for the field in the thermal state having an average of $\bar{n} = 20$ photons.

We can similarly study the bahaviour of any other observable. A property of particular interest is the squeezing of the field. The coherent field is classical. However, on interaction with the atom according to the Jaynes-Cummings hamiltonian, it exhibits the quantum feature of squeezing. For details, see [105].

Two-Channel Raman-Coupled Model. The hamiltonian in this case is given by (7.60). We let $g_s = g_A = g$. It corresponds to (11.1) with $h_0 = \delta = 0$, $\hat{F} = \hat{a}_p \hat{a}_S^\dagger + \hat{a}_p^\dagger \hat{a}_A$. Verify that

$$\hat{F}^\dagger \hat{F} = \hat{a}_p^\dagger \hat{a}_p \hat{a}_s \hat{a}_s^\dagger + \hat{a}_A^\dagger \hat{a}_A \hat{a}_p \hat{a}_p^\dagger + \hat{a}_p^{\dagger 2} \hat{a}_A \hat{a}_s + \hat{a}_s^\dagger \hat{a}_A^\dagger \hat{a}_p^2 \tag{11.44}$$

commutes with $\hat{M}$ and $\hat{C}$ where

$$\hat{M} = \hat{a}_p^\dagger \hat{a}_p + \hat{a}_s^\dagger \hat{a}_s + \hat{a}_A^\dagger \hat{a}_A, \quad \hat{C} = \hat{a}_A^\dagger \hat{a}_A - \hat{a}_s^\dagger \hat{a}_s. \tag{11.45}$$

Let $|m_p, m_A, m_s\rangle$ be an eigenstate of $\hat{a}_p^\dagger \hat{a}_p$ $\hat{a}_A^\dagger \hat{a}_A$ and $\hat{a}_s^\dagger \hat{a}_s$ with m_p, m_A, m_s as the corresponding eigenvalues. Owing to (11.45), the state space reduces to a sum of spaces labeled by the eigenvalues M and C of $\hat{M}$ and $\hat{C}$. Hence, we may employ the states $|M + C - 2m_s, m_s, m_s - C\rangle$ as a set of basis states. For the sake of illustration, let $m_s = m_A$, $M \to 2M$. An eigenstate $|\psi_\lambda\rangle$ of $\hat{F}^\dagger \hat{F}$ may then be expressed as

$$|\psi_\lambda\rangle = \sum_{m=0}^{M} C_m |2m, M - m, M - m\rangle. \tag{11.46}$$

Operate this with $\hat{F}^\dagger \hat{F}$ to obtain

$$\begin{aligned} &[2m(2M - 2m + 1) + M - m]\, C_m \\ &+(M - m)\sqrt{(2m + 1)(2m + 2)}\, C_{m+1} \\ &+(M - m + 1)\sqrt{2m(2m - 1)}\, C_{m-1} = \lambda C_m. \end{aligned} \tag{11.47}$$

Define

$$f_m = C_m/(M-m)!\sqrt{(2m)!}. \tag{11.48}$$

On substitution in (11.47) this yields

$$[2m(2M-2m+1)-m]\,f_m + (2m+1)(2m+2)f_{m+1} + (M-m+1)^2 f_{m-1} = (\lambda - M)f_m. \tag{11.49}$$

This is the same as (10.41) with

$$\begin{aligned} &\alpha_1 = 4M-3, \quad \alpha_2 = -4, \\ &\beta_0 = M^2, \quad \beta_1 = -2M+1, \quad \beta_2 = 1, \\ &\gamma_0 = 0, \quad \gamma_1 = 2, \quad \gamma_2 = 4. \end{aligned} \tag{11.50}$$

Verify that this falls in to the category C of classification of three-term recursion relations in Sect. 10.3.2. Note that the parameters (11.50) also satisfy (10.42) as they should because $C_{M+n} = 0$, $(n \geq 1)$. Invoke (10.62) to show that the eigenvalues are given by

$$\lambda = n(2n+1), \qquad n = 0, 1, \ldots, M. \tag{11.51}$$

The corresponding eigenvectors are also determined by (10.45). For some numerical results and their physical interpretation, see [84].

11.1.2 On-Resonant Real Time-Dependent Coupling

If $\delta = 0$ and g is real then verify that (11.13) are solved by

$$\begin{aligned} \hat{x}_{++} &= \hat{x}_{--} = -\mathrm{i}\frac{1}{\sqrt{\hat{N}}}\sin\left(\sqrt{\hat{N}}\tau\right), \\ \hat{x}_{+-} &= \hat{x}_{-+} = \cos\left(\sqrt{\hat{N}}\tau\right), \qquad \tau = \int g(t')\mathrm{d}t'. \end{aligned} \tag{11.52}$$

A realization of particular interest is $g(t) = a\sin(\omega t)$. This accounts for, for example, the effects of spatial mode structure on a moving atom in a cavity. This effect has been studied in [106] for single-mode two-level Jaynes-Cummings model. On inserting (11.52) in (11.10) we can recover not only the known analytic results [106] for the said model but can also study those effects in several other effective two-level models.

11.1.3 Fluctuating Coupling

We have thus at hand an apparatus for handling a two-level atom in a quantized field if the detuning and the coupling are deterministic functions of time. In this section we address the question of studying the dynamics when the said parameters are random functions of time.

In the case of fluctuating parameters, we first evaluate the expectation value $\langle \hat{A}(t) \rangle$ of an operator $\hat{A}$ and then average over the fluctuations. Note (for

example from (11.18)) that calculation of an expectation value involves action of $\hat{U}$ and $\hat{U}^\dagger$ together. Hence, the task of evaluating average over fluctuations reduces to that of finding that average over bilinear combinations of $\{\hat{x}_{ij}\}$. However, the operator whose expectation value is being evaluated, need not commute with $\hat{N}$. Hence, we need to know the average over fluctuations of the quantities of the type $x_{ij}(N_1)x_{kl}(N_2)$, $N_1 \neq N_2$. It may be determined by constructing equations for the said products.

As an example, apply (11.13) and (11.14) to show that

$$\begin{aligned} x_1 &= x_{++}(N_1)x^*_{++}(N_2), \quad x_2 = x_{++}(N_1)x^*_{-+}(N_2), \\ x_3 &= x_{-+}(N_1)x^*_{++}(N_2), \quad x_4 = x_{-+}(N_1)x^*_{-+}(N_2), \end{aligned} \tag{11.53}$$

obey the following closed set of equations

$$\begin{aligned} \mathrm{i}\dot{x}_1 &= -g(t)x_2 + g^*(t)x_3, \\ \mathrm{i}\dot{x}_2 &= -N_2 g^*(t)x_1 + \delta(t)x_2 + g^*(t)x_4, \\ \mathrm{i}\dot{x}_3 &= N_1 g(t)x_1 - \delta(t)x_3 - g(t)x_4, \\ \mathrm{i}\dot{x}_4 &= N_1 g(t)x_2 - N_2 g^*(t)x_3. \end{aligned} \tag{11.54}$$

Let us assume that δ is a constant and the coupling is of the form

$$g(t) = |g| \exp(-\mathrm{i}\phi(t)), \tag{11.55}$$

$|g|$ being a constant but the phase $\phi(t)$ a fluctuating variable. In order to find x_i averaged over fluctuations, define

$$\begin{aligned} x_{1m} &= \exp(\mathrm{i}m\phi(t))x_1, \quad x_{2m} = \exp(\mathrm{i}m\phi(t) - \mathrm{i}\phi(t))x_2, \\ x_{3m} &= \exp(\mathrm{i}m\phi(t) + \mathrm{i}\phi(t))x_3, \quad x_{4m} = \exp(\mathrm{i}m\phi(t))x_4. \end{aligned} \tag{11.56}$$

It then follows that

$$\bar{x}_1 = \bar{x}_{10}, \qquad \bar{x}_2 = \bar{x}_{21}, \qquad \bar{x}_3 = \bar{x}_{3-1}, \qquad \bar{x}_4 = \bar{x}_{40}. \tag{11.57}$$

On invoking (11.54), it is straightforward to show that

$$\begin{aligned} \mathrm{i}\begin{pmatrix} \dot{x}_{1m} \\ \dot{x}_{2m} \\ \dot{x}_{2m} \\ \dot{x}_{2m} \end{pmatrix} &= \dot{\phi}(t) \begin{pmatrix} m & 0 & 0 & 0 \\ 0 & m-1 & 0 & 0 \\ 0 & 0 & m+1 & 0 \\ 0 & 0 & 0 & m \end{pmatrix} \begin{pmatrix} x_{1m} \\ x_{2m} \\ x_{3m} \\ x_{4m} \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 & -|g| & |g| & 0 \\ -|g|N_2 & \delta & 0 & |g| \\ |g|N_1 & 0 & -\delta & -|g| \\ 0 & |g|N_1 & -|g|N_2 & 0 \end{pmatrix} \begin{pmatrix} x_{1m} \\ x_{2m} \\ x_{3m} \\ x_{4m} \end{pmatrix}. \end{aligned} \tag{11.58}$$

Recall from Chap. 6 that analytically exact equation for the average of the variables obeying an equation of the type (11.58) may be derived if $\dot{\phi}$ is either a delta-correlated Gaussian process or a random telegraph noise. We leave it to the reader to derive corresponding equations. The reader may also derive equations for other bilinear combinations of $x_\pm$.

11.2 N Two-Level Atoms in a Quantized Field

The Hamiltonian of N atoms interacting collectively with a quantized field may be written as in (11.1) with the understanding that $\hat{S}_\mu$ obey the $SU(2)$ commutation relations and that $S = N/2$ is the total spin quantum number. The space of the atomic states is then spanned by $N+1$ states $|m\rangle$ ($m = -S, -S+1, \ldots, S$. For the sake of illustration, let us assume that $\hat{F} = \hat{a}$. The combined state of the field and the atoms is then spanned by $|n, m\rangle$, n being the eigenvalue of $\hat{a}^\dagger\hat{a}$. Note that $\hat{M} = \hat{a}^\dagger\hat{a} + \hat{S}_z$ commutes with $\hat{H}$. Hence, if initial state of the system is such that $\hat{M}|\psi(0)\rangle = M|\psi(0)\rangle$, then its state at any time t will be a combination of states $|M-m, m\rangle$, where, the positivity of the photon number $M-m$ and the condition that $-S \leq m \leq S$ demand that $m = -S, -S+1, \ldots, M$ if $M \leq S$, but $m = -S, -S+1, \ldots, S$ if $M \geq S$. The eigenstates corresponding to the same eigenvalue M of $\hat{M}$ are said to constitute a *manifold*. Thus, the problem of N two-level atoms reduces to diagonalization of at most $N+1$ dimensional hamiltonian matrix. The dimension N_d is smaller than $N+1$ if $M \leq S$. For example, if $M = -S+1$ then $m = -S, -S+1$. The problem of N two-level atoms then reduces to solving the eigenvalue problem of only a two-dimensional matrix. Similar considerations apply if the process is multiphoton. For example, if $\hat{F} = \hat{a}^p$ in (11.1) then the constant opertator is $\hat{M} = \hat{a}^\dagger\hat{a} + p\hat{S}_z$.

11.3 Exactly Solvable Three-Level Systems

In this section we consider a class of systems consisting of three-level atoms in quantized field describable, in an appropriate rotating frame, by the hamiltonian

$$\hat{H}_2 = \hbar\Big[\delta_1(t)\hat{A}_{22} + \delta_2(t)\hat{A}_{33} + \Big\{g_1^*(t)\hat{A}_{21}\hat{F}_1 + g_2^*(t)\hat{A}_{32}\hat{F}_2 + \text{h.c.}\Big\}\Big]. \tag{11.59}$$

Here $\delta_1(t), \delta_2(t)$ are effective detunings, g_1, g_2 are the coupling constants,

$$\hat{A}_{ij} = |i\rangle\langle j|, \qquad i,j = 1,2,3 \tag{11.60}$$

are the operators effecting transition between the atomic states $|i\rangle$ and $|j\rangle$, and $\hat{F}_1, \hat{F}_2$ are the sums of products of single-mode quantized field operators. The orthogonality and completeness relations imply that

$$\hat{A}_{ij}\hat{A}_{kl} = \hat{A}_{il}\delta_{jk}, \qquad \sum_{i=1}^{3}\hat{A}_{ii} = 1. \tag{11.61}$$

As a consequence of this, the evolution operator $\hat{U}_2$ may be expressed as

$$\hat{U}_2 \equiv \exp\Big[-\frac{\mathrm{i}}{\hbar}\int_0^t \hat{H}_2(\tau)\mathrm{d}\tau\Big] = \sum_{i,j=1}^{3}\hat{\phi}_{ij}(t)\hat{A}_{ij}, \tag{11.62}$$

$\hat{\phi}_{ij}$ being functions of the field operators. In order to determine these functions, differentiate (11.62) with respect to time to get

$$\hat{H}_2\hat{U}_2 = \mathrm{i}\hbar \sum_{i,j=1}^{3} \dot{\hat{\phi}}_{ij}(t)\hat{A}_{ij}. \tag{11.63}$$

On substituting (11.59) for $\hat{H}_2$ and (11.62) for $\hat{U}_2$ and on comparing the coefficients of $\hat{A}_{ij}$, (11.63) leads to a closed set of equations

$$\begin{aligned}
\mathrm{i}\dot{\hat{\phi}}_{1i} &= g_1(t)\hat{F}_1^\dagger\hat{\phi}_{2i},\\
\mathrm{i}\dot{\hat{\phi}}_{2i} &= \delta_1(t)\hat{\phi}_{2i} + g_1^*(t)\hat{F}_1\hat{\phi}_{1i} + g_2(t)\hat{F}_2^\dagger\hat{\phi}_{3i},\\
\mathrm{i}\dot{\hat{\phi}}_{3i} &= \delta_2(t)\hat{\phi}_{3i} + g_2^*(t)\hat{F}_2\hat{\phi}_{2i}, \qquad i = 1,2,3.
\end{aligned} \tag{11.64}$$

These are to be solved with the initial condition

$$\hat{\phi}_{ij}(0) = 0 \quad \text{for} \quad i \neq j, \qquad \hat{\phi}_{11}(0) = \hat{\phi}_{22}(0) = \hat{\phi}_{33}(0) = 1. \tag{11.65}$$

Solving (11.64) analytically exactly is generally a formidable task. An analytically solvable case of widespread interest is when the δ's and the g's are time-independent with $\delta_2 = 0$. The equations (11.64) then yield

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\hat{\phi}_{2i} + \mathrm{i}\delta_1\frac{\mathrm{d}}{\mathrm{d}t}\hat{\phi}_{2i} + \hat{\Gamma}_{12}^2\hat{\phi}_{2i} = 0, \quad i = 1,2,3, \tag{11.66}$$

$$\hat{\Gamma}_{12}^2 = |g_1|^2\hat{F}_1\hat{F}_1^\dagger + |g_2|^2\hat{F}_2^\dagger\hat{F}_2. \tag{11.67}$$

The solution of (11.66) with the initial condition (11.65) is given by

$$\begin{aligned}
\hat{\phi}_{2i}(t) = \exp\Big(-\mathrm{i}\frac{\delta_1}{2}t\Big)\Big[\Big\{\cos(\hat{\Gamma}t) - \mathrm{i}\delta_1\frac{1}{2\hat{\Gamma}}\sin(\hat{\Gamma}t)\Big\}\delta_{i2}\\
-\mathrm{i}\frac{1}{\hat{\Gamma}}\sin(\hat{\Gamma}t)\Big\{g_1^*\hat{F}_1\delta_{i1} + g_2\hat{F}_2^\dagger\delta_{i3}\Big\}\Big],
\end{aligned} \tag{11.68}$$

δ_{ij} being the Kronecker delta, and

$$\hat{\Gamma}^2 = \frac{1}{4}\Big(\delta_1^2 + 4\hat{\Gamma}_{12}^2\Big). \tag{11.69}$$

The rest of the $\hat{\phi}_{ij}$, found on inserting (11.68) in (11.64), read

$$\hat{\phi}_{11} = 1 + |g_1|^2\hat{F}_1^\dagger\frac{1}{\hat{\Gamma}_{12}^2}\Big[\exp\Big(-\frac{\mathrm{i}\delta_1 t}{2}\Big)\Big(\cos(\hat{\Gamma}t) + \mathrm{i}\frac{\delta_1}{2\hat{\Gamma}}\sin(\hat{\Gamma}t)\Big) - 1\Big]\hat{F}_1, \tag{11.70}$$

$$\hat{\phi}_{31} = g_1^*g_2^*\hat{F}_2\frac{1}{\hat{\Gamma}_{12}^2}\Big[\exp\Big(-\frac{\mathrm{i}\delta_1 t}{2}\Big)\Big(\cos(\hat{\Gamma}t) + \mathrm{i}\frac{\delta_1}{2\hat{\Gamma}}\sin(\hat{\Gamma}t)\Big) - 1\Big]\hat{F}_1, \tag{11.71}$$

$$\hat{\phi}_{12} = -\mathrm{i}g_1\hat{F}_1^\dagger \frac{1}{\hat{\Gamma}} \exp(-\mathrm{i}\delta_1 t/2)\sin(\hat{\Gamma}t), \tag{11.72}$$

$$\hat{\phi}_{32} = -\mathrm{i}g_2^*\hat{F}_2 \frac{1}{\hat{\Gamma}} \exp(-\mathrm{i}\delta_1 t/2)\sin(\hat{\Gamma}t), \tag{11.73}$$

$$\hat{\phi}_{33} = 1 + |g_2|^2\hat{F}_2 \frac{1}{\hat{\Gamma}_{12}^2}\Big[\exp\Big(-\frac{\mathrm{i}\delta_1 t}{2}\Big)\Big(\cos(\hat{\Gamma}t) + \mathrm{i}\frac{\delta_1}{2\hat{\Gamma}}\sin(\hat{\Gamma}t)\Big) - 1\Big]\hat{F}_2^\dagger, \tag{11.74}$$

$$\hat{\phi}_{13} = g_1 g_2 \hat{F}_1^\dagger \frac{1}{\hat{\Gamma}_{12}^2}\Big[\exp(-\mathrm{i}\delta_1 t/2)\Big(\cos(\hat{\Gamma}t) + \mathrm{i}\frac{\delta_1}{2\hat{\Gamma}}\sin(\hat{\Gamma}t)\Big) - 1\Big]\hat{F}_2^\dagger. \tag{11.75}$$

We have thus determined completely the dynamics of a three-level system corresponding to the time-independent form of the Hamilonian (11.59) with $\delta_2 = 0$. For the standard models outlined in the Chap. 7, $\hat{F}_1$ and $\hat{F}_2$ are single mode operators. The corresponding $\hat{\Gamma}$ is a function of the number operators. In such cases, the number states constitute a convenient choice as a basis. For details of some numerical results, see [102].

Next, we discuss a frequently encountered situation in which a three-level hamiltonian is reducible to an effective two-level one.

11.4 Effective Two-Level Approximation

Note from the expressions for the $\{\hat{\phi}_{ij}\}$ derived in the last section that the rate of transition between the levels in the presence of the detuning δ_1 is determined by $||\Gamma||$ defined in (11.69). That rate in the absence of δ_1 is governed by $||\hat{\Gamma}_{12}||$. Now, let $|\delta_1| \gg ||\hat{\Gamma}_{12}||$. On ignoring the terms of order $|g_i|||\hat{F}_j||/||\Gamma_{12}||$, (11.68) and (11.70)-(11.75) reduce to

$$\hat{\phi}_{12} \approx \hat{\phi}_{21} \approx 0, \quad \hat{\phi}_{23} \approx \hat{\phi}_{32} \approx 0, \quad \hat{\phi}_{22} \approx \exp(-\mathrm{i}\delta_1 t), \tag{11.76}$$

$$\hat{\phi}_{11} \approx 1 + |g_1|^2\hat{F}_1^\dagger \frac{1}{\hat{\Gamma}_{12}^2}\Big[\exp\Big(\mathrm{i}\hat{\Gamma}_{12}^2 t/\delta_1\Big) - 1\Big]\hat{F}_1, \tag{11.77}$$

$$\hat{\phi}_{33} \approx 1 + |g_2|^2\hat{F}_2 \frac{1}{\hat{\Gamma}_{12}^2}\Big[\exp\Big(\mathrm{i}\hat{\Gamma}_{12}^2 t/\delta_1\Big) - 1\Big]\hat{F}_2^\dagger, \tag{11.78}$$

$$\hat{\phi}_{13} \approx g_1 g_2 \hat{F}_1^\dagger \frac{1}{\hat{\Gamma}_{12}^2}\Big[\exp\Big(\mathrm{i}\hat{\Gamma}_{12}^2 t/\delta_1\Big) - 1\Big]\hat{F}_2^\dagger, \tag{11.79}$$

$$\hat{\phi}_{31} \approx g_1^* g_2^* \hat{F}_2 \frac{1}{\hat{\Gamma}_{12}^2}\left[\exp\left(\mathrm{i}\hat{\Gamma}_{12}^2 t/\delta_1\right) - 1\right]\hat{F}_1. \tag{11.80}$$

The equations (11.76) show that, in the limit of large detuning, the levels $|1\rangle$ and $|3\rangle$ are decoupled from the level $|2\rangle$. Hence, if the atom is initially prepared in a superposition of only the states $|1\rangle$ and $|3\rangle$ then it continues executing transitions only between those two states. In other words, the three-level system then acts as an effective two-level one. The corresponding effective two-level hamiltonian can be written knowing the specific forms of the field operators $\hat{F}_1$ and $\hat{F}_2$. However, (11.77)-(11.80) on substitution in (11.62) determine the time-evolution in effective two-level approximation for arbitrary $\hat{F}_{1,2}$ without any recourse to the knowledge of the effective hamiltonian.

A frequently encountered three-level system is the one in which each of the two pairs of levels is coupled by only one mode. Consider, for example, the levels arranged in a ladder configuration with both the transitions induced by the same field mode described by the annihilation operator $\hat{a}$. It is described by (11.59) with $\hat{F}_1 = \hat{F}_2 = \hat{a}$. The operator solutions (11.77)-(11.80) in this case read

$$\hat{\phi}_{11} \approx 1 + |g_1|^2 \hat{a}^\dagger \hat{a} \frac{1}{\hat{\gamma}_1^2}\left[\exp\left(\mathrm{i}\hat{\gamma}_1^2 t/\delta_1\right) - 1\right], \tag{11.81}$$

$$\hat{\phi}_{33} \approx 1 + |g_2|^2 \hat{a}\hat{a}^\dagger \frac{1}{\hat{\gamma}_2^2}\left[\exp\left(\mathrm{i}\hat{\gamma}_2^2 t/\delta_1\right) - 1\right], \tag{11.82}$$

$$\hat{\phi}_{13} \approx g_1 g_2 \hat{a}^{\dagger 2} \frac{1}{\hat{\gamma}_2^2}\left[\exp\left(\mathrm{i}\hat{\gamma}_2^2 t/\delta_1\right) - 1\right], \tag{11.83}$$

$$\hat{\phi}_{31} \approx g_1^* g_2^* \hat{a}^2 \frac{1}{\hat{\gamma}_1^2}\left[\exp\left(\mathrm{i}\hat{\gamma}_1^2 t/\delta_1\right) - 1\right], \tag{11.84}$$

$$\begin{aligned}\hat{\gamma}_1^2 &= (|g_1|^2 + |g_2|^2)\hat{a}^\dagger \hat{a} - |g_2|^2, \\ \hat{\gamma}_2^2 &= (|g_1|^2 + |g_2|^2)\hat{a}^\dagger \hat{a} + 2|g_1|^2 + |g_2|^2.\end{aligned} \tag{11.85}$$

In writing these expressions, we have made use of (1.21). On combining (11.85) and (11.62) we obtain the operator determining the evolution of the levels $|1\rangle$ and $|3\rangle$ as

$$\hat{U}_{\text{eff}} = \exp\left[-\mathrm{i}\hat{H}_{\text{eff}}/\hbar\right] = \sum_{i,j=1,3} \hat{\phi}_{ij}|i\rangle\langle j|. \tag{11.86}$$

The $\hat{H}_{\text{eff}}$ satisfying (11.86) is given by

$$\begin{aligned}\hat{H}_{\text{eff}} = &\frac{1}{\delta_1}\left(|g_1|^2 \hat{a}^\dagger \hat{a}|1\rangle\langle 1| + |g_2|^2 \hat{a}\hat{a}^\dagger|3\rangle\langle 3|\right) \\ &+ \frac{1}{\delta_1}\left(g_1 g_2|3\rangle\langle 1|\hat{a}^2 + g_1^* g_2^*|1\rangle\langle 3|\hat{a}^{\dagger 2}\right).\end{aligned} \tag{11.87}$$

Note that this is the same as the one written in Sect. 7.5 on qualitative considerations.

In order to show that (11.87) is the correct hamiltonian, rewrite it in the form (11.1) and construct the corresponding evolution operator $\hat{U}_1$ as in (11.10). The correctness of (11.87) is established by showing that $\hat{U}_1$ so constructed is the same as the $\hat{U}_{\text{eff}}$ of (11.86). We leave it to the reader to carry the suggested steps.

The term in the first brackets in (11.87) is, as already stated in Chap. 7, the Stark shift arising due to virtual transitions to the intermediate level $|2\rangle$. In practice, it is common to ignore the Stark shift either completely or approximate it by replacing the field operator $\hat{a}\hat{a}^\dagger$ in front of the atomic operator $|3\rangle\langle 3|$ in (11.87) by $\hat{a}^\dagger\hat{a}$. This replacement amounts to ignoring spontaneous emission from the upper level. However, the $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$ in the corresponding $\hat{U}_1(t)$ of (11.21) in these cases turn out to be square roots of an imperfect quadratic form in the number operator $\hat{a}^\dagger\hat{a}$. This implies irrational ratio of frequencies in the number states. Correct frequencies (11.85) are, however, linear in the number operator $\hat{a}^\dagger\hat{a}$. The expressions of $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$ in agreement with (11.85) are obtained, as already established above, by treating the Stark shift correctly. For some numerical results in case the Stark shift is treated approximately, see [107]- [109]. The reference [109] compares the results obtained in the effective two-level approximation with the exact results.

The effective Hamiltonians for the three levels in other configurations and in two-mode field may similarly be derived and shown to be the ones given in Chap. 7.

12. Dissipative Atomic Systems

In the last chapter we outlined an approach to studying lossless two- and three-level atomic systems interacting with lossless quantized field. In this chapter we analyze the equations governing dissipative two and three level atomic systems driven by an external field.

12.1 Two-Level Atom in a Quasimonochromatic Field

Recall from Chap. 8 that the dynamical evolution of the density operator of a two-level atom in an external field is governed by the master equation

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = -\frac{\mathrm{i}}{\hbar}\left[\hat{H}_{\mathrm{ext}},\ \hat{\rho}\right] - \mathrm{i}\omega_0\left[\hat{S}_z,\ \hat{\rho}\right] + \hat{\hat{L}}_{\mathrm{r}}\hat{\rho} + \hat{\hat{L}}_{\mathrm{c}}\hat{\rho}. \tag{12.1}$$

Here $\hat{H}_{\mathrm{ext}}$ is the hamiltonian of interaction between the atoms and an external field, assumed to be of the form

$$\hat{H}_{\mathrm{ext}} = \hbar\left[g(t)\hat{S}_+ \exp(-\mathrm{i}\omega_{\mathrm{l}}t) + g^*(t)\hat{S}_- \exp(\mathrm{i}\omega_{\mathrm{l}}t)\right]. \tag{12.2}$$

The Liouvillean $\hat{\hat{L}}_r$ in (12.1) describes the atomic radiative losses. It is given by (8.95). The Liouvillean $\hat{\hat{L}}_c$ governs the losses due to atomic collisions. It is given by (8.106). We transform to the interaction picture by means of

$$\hat{\rho}_{\mathrm{I}}(t) = \exp(\mathrm{i}\omega_{\mathrm{l}}\hat{S}_z t)\hat{\rho}(t)\exp(-\mathrm{i}\omega_{\mathrm{l}}\hat{S}_z t). \tag{12.3}$$

The atomic dynamics in this picture is characterized by

$$\begin{aligned}
\dot{\hat{\rho}}_{\mathrm{I}} = &-\mathrm{i}\Big[\delta\hat{S}_z + g(t)\hat{S}_+ + g^*(t)\hat{S}_-,\ \hat{\rho}_{\mathrm{I}}\Big] \\
&+\gamma\Big[(\bar{N}+1)\Big(2\hat{S}_-\hat{\rho}_{\mathrm{I}}\hat{S}_+ - \hat{\rho}_{\mathrm{I}}\hat{S}_+\hat{S}_- - \hat{S}_+\hat{S}_-\hat{\rho}_{\mathrm{I}}\Big) \\
&+\bar{N}\Big(2\hat{S}_+\hat{\rho}_{\mathrm{I}}\hat{S}_- - \hat{\rho}_{\mathrm{I}}\hat{S}_-\hat{S}_+ - \hat{S}_-\hat{S}_+\hat{\rho}_{\mathrm{I}}\Big) \\
&+\Big\{\bar{M}\exp(2\mathrm{i}\delta_p t)\Big(2\hat{S}_-\hat{\rho}_{\mathrm{I}}\hat{S}_- - \hat{\rho}_{\mathrm{I}}\hat{S}_-\hat{S}_- - \hat{S}_-\hat{S}_-\hat{\rho}_{\mathrm{I}}\Big) + \mathrm{h.c.}\Big\} \\
&+\gamma_{\mathrm{c}}\Big(2\hat{S}_z\hat{\rho}_{\mathrm{I}}\hat{S}_z - \hat{\rho}_{\mathrm{I}}\hat{S}_z\hat{S}_z - \hat{S}_z\hat{S}_z\hat{\rho}_{\mathrm{I}}\Big)\Big].
\end{aligned} \tag{12.4}$$

Here $\delta = \omega_0 - \omega_l$ is the detuning of the atomic levels from the frequency of the driving field, $\delta_p = \Omega_p - \omega_l$ is the difference in the frequency between the driving field and that of the pump driving the squeezed bath.

Invoke (8.37) and the properties of two-level operators to show that the atomic averages obey the equation (with $\bar{M} = |\bar{M}| \exp(i\phi)$)

$$\frac{d}{dt}|S(t)\rangle = \hat{M}|S(t)\rangle - \gamma \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{12.5}$$

$$|S(t)\rangle = \begin{pmatrix} \langle \hat{S}_x(t) \rangle \\ \langle \hat{S}_y(t) \rangle \\ \langle \hat{S}_z(t) \rangle \end{pmatrix}, \tag{12.6}$$

$$\hat{M}(t) = \begin{pmatrix} -\Gamma_{ph} + \Gamma_C & \Gamma_S - \delta & 0 \\ \Gamma_S + \delta & -\Gamma_{ph} - \Gamma_C & -2g(t) \\ 0 & 2g(t) & -\Gamma \end{pmatrix}. \tag{12.7}$$

$$\begin{aligned} &\Gamma = 2\left(2\bar{N} + 1\right)\gamma, \qquad \Gamma_{ph} = \frac{\Gamma}{2} + \gamma_c, \\ &\Gamma_C = 2|\bar{M}|\gamma \cos(\phi + \delta_p\, t), \qquad \Gamma_S = 2|\bar{M}|\gamma \sin(\phi + \delta_p\, t). \end{aligned} \tag{12.8}$$

The coupling constant $g(t)$ has been taken to be real for convenience.

Note from (8.67) that $\bar{N}(\bar{N}+1) \geq |\bar{M}|^2$. This implies that $(2\bar{N}+1)^2 - 4|\bar{M}|^2 = 4\bar{N}(\bar{N}+1) - 4|\bar{M}|^2 + 1 \geq 0$, i.e., $2\bar{N} + 1 - 2|\bar{M}| \geq 0$. As a consequence of this we have

$$\Gamma_{ph} - 2|\bar{M}|\gamma \geq 0. \tag{12.9}$$

The equations (12.5) are the so called optical *Bloch equations*. Their formal solution evidently is

$$|S(t)\rangle = \hat{D}(t,0)|S(0)\rangle - \gamma \int_0^t \hat{D}(t,\tau)d\tau \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{12.10}$$

where $\hat{D}(t,t')$ is the time-ordered integral

$$\hat{D}(t,t') = \overleftarrow{T} \exp\left(\int_{t'}^{t} \hat{M}(\tau)d\tau\right). \tag{12.11}$$

This integral is easily evaluated if $\hat{M}$ is time-independent (see Sect. 12.1.2). However, its explicit evaluation for general time-dependence of $\hat{M}(t)$ is not straightforward. The complexity of the problem, and its possible simplifications, can be assessed by rewriting $\hat{M}(t)$ in terms of nine elementary matrices

$$\hat{B}_{pq} = |p\rangle\langle q|, \qquad p, q = 1, 2, 3, \tag{12.12}$$

the $|p\rangle$ being an elementary column vector which has unity as its only non-zero element in its p^{th} row. Consequently, $\hat{B}_{pq}$ has unity as its only non-zero element at the position (p, q). Clearly

$$\begin{aligned}\hat{M}(t) = {} & 2g(t)\left(\hat{B}_{32} - \hat{B}_{23}\right) + \delta\left(\hat{B}_{21} - \hat{B}_{12}\right) - \Gamma_{\text{ph}}\left(\hat{B}_{11} + \hat{B}_{22}\right) \\ & -\Gamma\hat{B}_{33} + 2|\bar{M}|\gamma\Big\{\sin(\phi + \phi_p t)\left(\hat{B}_{12} + \hat{B}_{21}\right) \\ & + \cos(\phi + \phi_p t)\left(\hat{B}_{11} - \hat{B}_{22}\right)\Big\}. \end{aligned} \tag{12.13}$$

Analytically tractable cases of the time-ordered integral (12.11) can be identified by looking for the cases in which $\hat{M}(t)$ reduces to a combination of a set of a fewer number of operators closed under commutation. For example, it can be verified that $\hat{B}_{12}$, $\hat{B}_{21}$ $(1/2)(\hat{B}_{11} - \hat{B}_{22})$ obey the commutation relations of the $SU(2)$ operators $\hat{J}_{\pm}$, $\hat{J}_z$. We may similarly identify other sets of operators obeying $SU(2)$. Now, if the values of the parameters are such that $\hat{M}(t)$ reduces to such combinations of $\hat{B}_{ij}$ which are closed under $SU(2)$ then the problem of evaluating (12.11) is reduced to that of finding an $SU(2)$ time-ordered integral. That problem has already been addressed in Chap. 2. We discuss in the subsection below the dynamics when $\hat{M}$ is time-dependent and reducible to $SU(2)$. In the subsection following it, we discuss the case of time-independent $\hat{M}$.

12.1.1 Time-Dependent Evolution Operator Reducible to $SU(2)$

The operator $\hat{M}(t)$ reduces to an $SU(2)$ operator if: (a) there is no damping i.e. if $\gamma = \gamma_c = 0$; or (b) if $\Gamma_{\text{ph}} = \Gamma$, $|\bar{M}| = 0$; or (c) if $\delta = \delta_p = 0$, along with $\phi = n\pi$. In the following we discuss these cases separately.

No Damping. In this case, the equation of evolution (12.5) for the averages is homogeneous. The expression (12.13) for $\hat{M}$ reads

$$\begin{aligned}\hat{M}(t) &= 2g(t)\left(\hat{B}_{32} - \hat{B}_{23}\right) + \delta\left(\hat{B}_{21} - \hat{B}_{12}\right) \\ &\equiv -\mathrm{i}\Big(2g(t)\hat{J}_x + \delta\hat{J}_y\Big) \end{aligned} \tag{12.14}$$

where, as can be readily verified, the operators

$$\hat{J}_x = \mathrm{i}(\hat{B}_{32} - \hat{B}_{23}), \quad \hat{J}_y = \mathrm{i}(\hat{B}_{21} - \hat{B}_{12}), \quad \hat{J}_z = \mathrm{i}(\hat{B}_{31} - \hat{B}_{13}), \tag{12.15}$$

obey the $SU(2)$ commutation relations. They are the generators of its three-dimensional representation having the property $\hat{J}_\mu^3 = \hat{J}_\mu$, the $\hat{J}_\mu$ being any component of $\hat{\boldsymbol{J}}$. The expression (12.11) then reads

$$\hat{D}(t, t') = \overleftarrow{T} \exp\left[-\mathrm{i}\int_{t'}^{t}\Big(2g(\tau)\hat{J}_x + \delta\hat{J}_y\Big)\,\mathrm{d}\tau\right]. \tag{12.16}$$

The time-integral can now be performed by using the method of Sect. 2.5.2. The time-ordered integral in (12.16) reduces to the exponential of the integral if $\delta = 0$. In this case, apply (2.14) to show that

$$\hat{D}(t,t') = 1 - \mathrm{i}\sin\left(2\theta(t,t')\right)\hat{J}_x + \left\{\cos\left(2\theta(t,t')\right) - 1\right\}\hat{J}_x^2, \tag{12.17}$$

$$\theta(t,t') = \int_{t'}^{t} g(\tau)\mathrm{d}\tau. \tag{12.18}$$

Since $g(t)$ is proportional to the envelop function of the driving field, $\theta(t,t')$ is proportional to the area enclosed between the field envelop and the time axis between the times t' and t. Substitution of (12.17) in (12.10) along with the use of matrix representation of $\hat{J}_x$, yields

$$\begin{pmatrix} \langle\hat{S}_x(t)\rangle \\ \langle\hat{S}_y(t)\rangle \\ \langle\hat{S}_z(t)\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\left(2\theta(t,0)\right) & -\sin\left(2\theta(t,0)\right) \\ 0 & \sin\left(2\theta(t,0)\right) & \cos\left(2\theta(t,0)\right) \end{pmatrix} \times \begin{pmatrix} \langle\hat{S}_x(0)\rangle \\ \langle\hat{S}_y(0)\rangle \\ \langle\hat{S}_z(0)\rangle \end{pmatrix}. \tag{12.19}$$

This determines the dynamics of a resonantly driven non-dissipating two-level atom in a time-dependent field. Now, let the atom be initially in one of the two states $|\pm 1/2\rangle$ so that $\langle\hat{S}_x(0)\rangle = \langle\hat{S}_y(0)\rangle = 0, \langle\hat{S}_z(0)\rangle = \pm 1/2$. It will be found in the other state $|\mp 1/2\rangle$ so that $\langle\hat{S}_x(t)\rangle = \langle\hat{S}_y(t)\rangle = 0, \langle\hat{S}_z(t)\rangle = \mp 1/2$ at the time t which is such that the area $\theta(t,0) = (2n+1)\pi/2$. It will return to its initial state at such time t which generates the area $\theta(t,0) = n\pi$.

Strong Collisions and Thermal Bath. Consider the case $\Gamma_{\mathrm{ph}} = \Gamma$ and $|\bar{M}| = 0$. The condition $|\bar{M}| = 0$ means that the reservoir is thermal. The condition $\Gamma_{\mathrm{ph}} = \Gamma$, by virtue of (12.8), requires collisions to be so strong that $\gamma_c = \Gamma/2$. This condition is, therefore, referred to as the *strong collisions limit*. Since, by definition (12.12), $\hat{B}_{11} + \hat{B}_{22} + \hat{B}_{33} = \hat{I}$, the expression (12.13) for $\hat{M}(t)$ in this case reduces to

$$\begin{aligned} \hat{M}(t) &= 2g(t)\left(\hat{B}_{32} - \hat{B}_{23}\right) + \delta\left(\hat{B}_{21} - \hat{B}_{12}\right) - \Gamma I \\ &\equiv -\mathrm{i}\Big(2g(t)\hat{J}_x + \delta\hat{J}_y\Big) - \Gamma I, \end{aligned} \tag{12.20}$$

where $\hat{J}_\mu$ are as in (12.15). Hence, (12.11) may be rewritten as

$$\hat{D}(t,t') = \exp\left(-\Gamma\, t\right)\overleftarrow{T}\exp\left[-\mathrm{i}\int_{t'}^{t}\Big(2g(\tau)\hat{J}_x + \delta\hat{J}_y\Big)\,\mathrm{d}\tau\right]. \tag{12.21}$$

The time-ordered integral above describes free evolution. The solution of the homogeneous part of (12.5) in the limit of strong collisions is thus a product of free and damped evolutions.

Resonant Squeezed Bath. Consider the case $\delta = \delta_p = 0$, $\phi = n\pi$. This implies that $\Gamma_S = 0$, $\Gamma_C = 2(-)^n|\bar{M}|\gamma$. Note that in this case the equation for $\langle\hat{S}_x(t)\rangle$ reading

$$\langle\dot{\hat{S}}_x(t)\rangle = -\left(\Gamma_{\mathrm{ph}} - \Gamma_C\right)\langle\hat{S}_x(t)\rangle \tag{12.22}$$

gets decoupled from the equations for $\langle \hat{S}_y(t) \rangle$ and $\langle \hat{S}_z(t) \rangle$:

$$\begin{aligned} \langle \dot{\hat{S}}_y(t) \rangle &= -\left[\Gamma_{\rm ph} + \Gamma_C\right] \langle \hat{S}_y(t) \rangle - 2g(t) \langle \hat{S}_z(t) \rangle \\ \langle \dot{\hat{S}}_z(t) \rangle &= -\Gamma \langle \hat{S}_z(t) \rangle + 2g(t) \langle \hat{S}_y(t) \rangle . \end{aligned} \tag{12.23}$$

If g is independent of time then these equations can be solved by extending the results of the Appendix C. Else, they may be handled by the method of time-ordered $SU(2)$ integration outlined in Sect. 2.5.2.

12.1.2 Time-Independent Evolution Operator

If the envelop of the driving field is constant and its frequency the same as that of the pump driving the squeezed bath then all the elements of $\hat{M}$ become time-independent. The master equation then describes the phenomenon of resonance fluorescence in a squeezed bath. The evolution operator $\hat{M}$ in this case assumes the form

$$\hat{M} = \begin{pmatrix} -\Gamma_{\rm ph} + \Gamma_1 & \Gamma_2 - \delta & 0 \\ \Gamma_2 + \delta & -\Gamma_{\rm ph} - \Gamma_1 & -2g \\ 0 & 2g & -\Gamma \end{pmatrix}, \tag{12.24}$$

$$\Gamma_1 = 2|\bar{M}|\gamma \cos(\phi), \qquad \Gamma_2 = 2|\bar{M}|\gamma \sin(\phi), \tag{12.25}$$

and $\Gamma_{\rm ph}$ is as in (12.8). The formal solution (12.10) now reads

$$|S(t)\rangle = \exp\left(\hat{M}t\right) |S(0)\rangle - \gamma \int_0^t \exp\left(\hat{M}\tau\right) {\rm d}\tau \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{12.26}$$

This may be evaluated by applying (C.7). Its application requires eigenvalues of $\hat{M}$. It is straightforward to verify that the equation $\det(\hat{M} - \lambda I) = 0$ determining the eigenvalues λ is the cubic

$$\begin{aligned} f(\lambda) &= \left[(\lambda + \Gamma_{\rm ph})^2 - \Gamma_1^2 - \Gamma_2^2 + \delta^2\right] [\lambda + \Gamma] + 4g^2 \left[\lambda + \Gamma_{\rm ph} - \Gamma_1\right] \\ &\equiv \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0, \end{aligned} \tag{12.27}$$

with

$$\begin{aligned} a_0 &= \Gamma \left(\Gamma_{\rm ph}^2 - 4|\bar{M}|^2\gamma^2 + \delta^2\right) + 4g^2 \left(\Gamma_{\rm ph} - \Gamma_1\right), \\ a_1 &= 2\Gamma_{\rm ph}\Gamma + \Gamma_{\rm ph}^2 - 4|\bar{M}|^2\gamma^2 + \delta^2 + 4g^2, \\ a_2 &= \Gamma + 2\Gamma_{\rm ph}. \end{aligned} \tag{12.28}$$

Let λ_i $(i = 1, 2, 3)$ be the roots of (12.27). We invoke (C.7) to evaluate $\exp(\hat{M}t)$ and insert it in (12.26). We get

$$
\begin{aligned}
\langle \hat{S}_x(t) \rangle &= \left[\alpha_1 + \alpha_2 (\Gamma_1 - \Gamma_{\rm ph}) + \alpha_3 \left\{ (\Gamma_1 - \Gamma_{\rm ph})^2 + \Gamma_2^2 - \delta^2 \right\}\right] \langle \hat{S}_x(0) \rangle \\
&\quad + (\Gamma_2 - \delta)(\alpha_2 - 2\alpha_3 \Gamma_{\rm ph}) \langle \hat{S}_y(0) \rangle - 2g\alpha_3 (\Gamma_2 - \delta) \langle \hat{S}_z(0) \rangle, \\
&\quad + 2\gamma g (\Gamma_2 - \delta) \int_0^t \alpha_3(\tau) \mathrm{d}\tau, \\
\langle \hat{S}_y(t) \rangle &= (\Gamma_2 + \delta)(\alpha_2 - 2\alpha_3 \Gamma_{\rm ph}) \langle \hat{S}_x(0) \rangle \\
&\quad + \Big[\alpha_1 - \alpha_2 (\Gamma + \Gamma_{\rm ph}) \\
&\quad + \alpha_3 \left\{ (\Gamma_1 + \Gamma_{\rm ph})^2 + \Gamma_2^2 - \delta^2 - 4g^2 \right\}\Big] \langle \hat{S}_y(0) \rangle \\
&\quad - 2g \left[\alpha_2 - \alpha_3 (\Gamma_1 + \Gamma_{\rm ph} + \Gamma)\right] \langle \hat{S}_z(0) \rangle \\
&\quad + 2\gamma g \int_0^t \left[\alpha_2(\tau) - \alpha_3(\tau)(\Gamma_1 + \Gamma_{\rm ph} + \Gamma)\right] \mathrm{d}\tau, \\
\langle \hat{S}_z(t) \rangle &= 2g\alpha_3 (\Gamma_2 + \delta) \langle \hat{S}_x(0) \rangle + 2g \left[\alpha_2 - \alpha_3 (\Gamma_1 + \Gamma_{\rm ph} + \Gamma)\right] \langle \hat{S}_y(0) \rangle \\
&\quad + \left[\alpha_1 - \Gamma\alpha_2 + \alpha_3 \left(\Gamma^2 - 4g^2\right)\right] \langle \hat{S}_z(0) \rangle \\
&\quad - \gamma \int_0^t \left[\alpha_1(\tau) - \Gamma\alpha_2(\tau) + \alpha_3(\tau) \left(\Gamma^2 - 4g^2\right)\right] \mathrm{d}\tau. \qquad (12.29)
\end{aligned}
$$

This gives the atomic averages in terms of $\{\alpha_m(t)\}$ which are determined by evaluating (C.8) in terms of the roots of (12.27).

The problem of studying the radiative properties of a two-level atom in a monochromatic field thus boils down to the one of finding the roots of the cubic in (12.27). Exact analytic expression for the roots of a cubic are known and are reproduced in the Appendix D. However, as elaborated below, the nature of the roots and hence the qualitative features of the evolution can be established even without solving (12.27) explicitly.

Recall from the Appendix D that the roots of a cubic are either all real or one of the roots is real and the other two are complex conjugates of each other. We show that the real part of each of the roots of (12.27) is negative. To that end, we invoke the Hurwitz criterion stated in the Appendix D and note that the roots have a negative real part if (a) $a_0, a_1 > 0$, and (b) $a_1 a_2 > a_0$. Using the definitions (12.28) of the $a_i's$ and the condition (12.9), verify that both these conditions are satisfied for any non-zero value of γ and γ_c. As a consequence of this it follows that, as $t \to \infty$, the atom evolves irreversibly to a steady state. That state is determined by (C.3).

Now, following again the Appendix D, we express the roots in terms of one unknown parameter lying in the range [0,1]. This approach reveals easily the conditions under which a pair of roots become complex. To that end, note that if $\Gamma/2 - \gamma_c + \Gamma_1 > 0$ then $f(\lambda)$ of (12.27) is such that

$$
f(-\Gamma) = -4g^2 \left[\frac{\Gamma}{2} - \gamma_c + \Gamma_1\right] \leq 0,
$$

$$f\left(-\Gamma_{\rm ph}+2|\bar{M}|\gamma\right)=\delta^2\left[\frac{\Gamma}{2}+2|\bar{M}|\gamma-\gamma_c\right]$$
$$+8g^2|\bar{M}|\gamma(1-\cos(\phi))\geq 0. \tag{12.30}$$

Hence, $f(\lambda)$ has a root, say λ_1, in the interval $[-\Gamma_{\rm ph}+2|\bar{M}|\gamma,\ -\Gamma]$. We note that, due to (12.9), $-\Gamma_{\rm ph}+2|\bar{M}|\gamma<0$. Hence $\lambda_1<0$. We may write λ_1 as

$$\lambda_1=(2|\bar{M}|\gamma-\Gamma_{\rm ph})(1-\alpha)-\Gamma\alpha,\qquad 0\leq\alpha\leq 1. \tag{12.31}$$

The other two roots, $\lambda_{2,3}$, are then the solution of the quadratic (D.5). These roots will be complex if the discriminant of the quadratic is negative, i.e. if

$$(\lambda_1+a_2)^2-4\left[a_1+\lambda_1(\lambda_1+a_2)\right]<0. \tag{12.32}$$

This, on inserting the $a_i's$ from (12.28), acquires the form

$$D^2-\Omega^2(\delta)<0 \tag{12.33}$$

where D is a combination of the damping constants and

$$\Omega(\delta)=\sqrt{4g^2+\delta^2} \tag{12.34}$$

is the off-resonance Rabi frequency. This shows that complex roots are obtained above a threshold value of the Rabi frequency.

We have thus at hand a complete description of the atomic dynamics. As discussed in Chap. 8, the atomic averages determine also the statistical properties of the radiation from the atom. We have in (8.76) the expression for the field in the radiation zone. It shows that the positive frequency part of the field at time time t and at the position $\boldsymbol{r}$ with respect to the atom is proportional to $\hat{S}_-(t-|\boldsymbol{r}|/c)$. Hence, the problem of finding the average of a normal order product of the field operators reduces to that of determining the product of atomic operator in which $\hat{S}_-$ are placed at the right of the $\hat{S}_+$. Of particular interest are the first-order and second-order coherence functions of the field. These are proportional respectively to

$$G^{(1)}(\tau)=\langle\hat{S}_+(t+\tau)\hat{S}_-(t)\rangle,$$
$$G^{(2)}(\tau)=\langle\hat{S}_+(t)\hat{S}_+(t+\tau)\hat{S}_-(t+\tau)\hat{S}_-(t)\rangle. \tag{12.35}$$

Note that because of the two-level property $\hat{S}_\pm^2=0$, $G^{(2)}(0)=0$. This implies that the radiation from a two-level atom is antibunched.

The multi-time averages may be evaluated by invoking the regression theorems (Chap. 8). To that end, let

$$\langle\hat{S}_+(t)\rangle=f_{+0}(t)+\sum_{i=\pm,z}f_{+i}(t)\langle\hat{S}_i(0)\rangle,$$
$$\langle\hat{S}_+(t)\hat{S}_-(t)\rangle=\frac{1}{2}+\langle\hat{S}_z(t)\rangle=\frac{1}{2}+f_{z0}(t)+\sum_{i=\pm,z}f_{zi}(t)\langle\hat{S}_i(0)\rangle. \tag{12.36}$$

The $f(t)'s$ can be identified by comparing these expressions with (12.29). On using (8.47) and the properties of two-level operators it follows that

$$G^{(1)}(\tau) = f_{+0}(\tau)\langle\hat{S}_-(t)\rangle + f_{++}(\tau)\left(\langle\hat{S}_z(t)\rangle + \frac{1}{2}\right)$$
$$-\frac{1}{2}f_{+z}(\tau)\langle\hat{S}_-(t)\rangle,$$
$$G^{(2)}(\tau) = \left(\frac{1}{2} + f_{z0}(\tau)\right)\langle\hat{S}_+(t)\hat{S}_-(t)\rangle + \sum_{i=\pm,z} f_{zi}(\tau)\langle\hat{S}_+(t)\hat{S}_i(t)\hat{S}_-(t)\rangle$$
$$= \left(\frac{1}{2} + f_{z0}(\tau) - \frac{1}{2}f_{zz}(\tau)\right)\left(\langle\hat{S}_z(t)\rangle + \frac{1}{2}\right). \tag{12.37}$$

Consider a damped atom so that it reaches a steady state. Evaluate the correlation functions in the limit $t \to \infty$. The corresponding averages are found using (C.3). In the following we highlight some qualitative features of the emission and the absorption spectra obtained by employing the steady state atomic correlation functions.

Consider first the emission spectrum. The spectrum is defined in (6.91). We substitute in it the expression (8.75) for the field from the atom in the radiation zone. We assume the free-field to be in the state of the vacuum. We also note that we have evaluated the averages in the interaction picture defined in (12.3). It is in a frame rotating at the driving field frequency ω_l. Hence, the expression (6.91) for the spectrum in the interaction picture, with the field given by (8.75), acquires the form

$$S(\omega) = \mathrm{Re}\int_0^\infty \exp\{-\mathrm{i}(\omega-\omega_l)\tau\}G(\tau)\mathrm{d}\tau. \tag{12.38}$$

This gives the spectrum of the field from the driven atom. It is, therefore, also referred to as the *emission spectrum*. This involves evaluating expressions of the form

$$I(\omega) = \mathrm{Re}\int_0^\infty \mathrm{d}\tau \exp\{-\mathrm{i}(\omega-\omega_l)\tau\}\exp\left\{(-\lambda_R + \mathrm{i}\lambda_I)\,\tau\right\}, \tag{12.39}$$

λ_R and λ_I being the real and imaginary parts of an eigenvalue λ. Note that there also are time-independent terms in the expressions for the averages and consequently τ-independent terms in $G(\tau)$. They correspond to $\lambda_R = \lambda_I = 0$. $I(\omega)$ due to such terms is the delta function $\delta(\omega-\omega_l)$. This component in the spectrum is called the *Rayleigh or coherent component*. It represents elastic scattering of radiation at its incident frequency. For τ-dependent terms in $G(\tau)$, (12.39) yields

$$I(\omega) \sim \frac{\lambda_R}{\lambda_R^2 + (\omega-\omega_l+\lambda_I)^2}. \tag{12.40}$$

This is a Lorentzian of width λ_R centered at $\omega = \omega_l - \lambda_I$. A real root contributes a Lorentzian at the frequency of the driving field whereas a pair of complex conjugate roots contribute Lorentzians at $\omega_l \pm \lambda_I$ in the spectrum. A Lorentzian characterizes an incoherent process. It results due to absorption of

a photon from the incident field followed by the process of spontaneous emission. Recall that the roots become complex if the field is sufficiently strong. The splitting of the fluorescent spectrum in to three components in a strong field is referred to as *ac Stark splitting.*

We may similarly determine the absorption spectrum by evaluating (9.26). In the interaction picture it assumes the form

$$W_a(\omega) \sim \mathrm{Re} \int_0^\infty \exp\{\mathrm{i}(\omega - \omega_1)\tau\} \left\langle \left[\hat{S}_-(\tau), \hat{S}_+\right] \right\rangle \mathrm{d}\tau. \tag{12.41}$$

It consists of a Lorentzian centred at the atomic transition frequency if the field intensity is below its threshold value for the complex roots to occur. Lorentzian peaks displaced from the centre by $\pm\lambda_1$ appear in a strong field.

The qualitative considerations outlined above, however, do not provide information about the relative heights of the peaks. The roots need to be determined explicitly for a quantitative study. As stated before, the analytic expression for the roots is generally not simple. It assumes simple form whenever the cubic can be factored in to the product of a quadratic and a linear form in λ. Such special situations are encountered if (c1) the atom is undamped; (c2) the atom is not driven, i.e. $g = 0$, (c3) $|\bar{M}| = 0$ and the collisions are strong so that $\Gamma_{\mathrm{ph}} = \Gamma$, (c4) the drive is on-resonance, i.e. $\delta = 0$, and $\phi = n\pi$. Construction of solution in these cases is a matter of simple algebra left for the reader to carry. Simple results are obtained also when the drive is strong, i.e. $\Omega(\delta) \gg \gamma$. This case is discussed in the next section for a system of N two-level atoms.

12.1.3 Nonlinear Response in a Bichromatic Field

A problem of considerable interest is the nonlinear response of a two-level atom in a bichromatic field consisting of frequencies ω_1, ω_2. This interaction is characterized by

$$\hat{H}_{\mathrm{ext}} = \hbar \left[\epsilon_1 \exp(-\mathrm{i}\omega_1 t)\hat{S}_+ + \epsilon_2 \exp(-\mathrm{i}\omega_2 t)\hat{S}_+ + \mathrm{h.c.}\right]. \tag{12.42}$$

We assume that $\hat{H}_{\mathrm{ext}}$ may be treated perturbatively and that $|\epsilon_1| > |\epsilon_2|$. We study the response to second order in ϵ_1 and first order in ϵ_2. We are interested in the characteristics of the four-wave mixing signal at the frequency $\Omega = 2\omega_1 - \omega_2$. It is characterized by $\chi^{(3)}(\omega_1, \omega_1, -\omega_2)$. We construct it using the definition (9.18). The summation in that formula runs over $\nu_{i_1}, \nu_{i_2}, \nu_{i_3}$. In the present situation, each of these frequencies takes values $\omega_1, \omega_1, -\omega_2$. The distinct combinations of $(\nu_{i_1}, \nu_{i_2}, \nu_{i_3})$ are $(\omega_1, \omega_1, -\omega_2)$, $(\omega_1, -\omega_2, \omega_1)$ and $(-\omega_2, \omega_1, \omega_1)$. We recall also that $\hat{\hat{L}}(\pm\nu)\hat{F} \sim -\mathrm{i}[\hat{S}_\mp, \hat{F}]$. Consequently, it follows that

$$\chi^{(3)}(\omega_1, \omega_1, -\omega_2) \sim \mathbf{Tr}[\hat{S}_+ \rho^{(3)}(\Omega)], \tag{12.43}$$

$$\rho^{(3)}(\Omega) \sim \frac{1}{\hat{L} - \mathrm{i}\Omega}\Big\{[S_-, \frac{1}{\hat{\hat{L}} - \mathrm{i}(\omega_1 - \omega_2)}[S_-, \frac{1}{\hat{\hat{L}} + \mathrm{i}\omega_2}[S_+, \rho_{\mathrm{ss}}]]]$$
$$+ [S_-, \frac{1}{\hat{\hat{L}} - \mathrm{i}(\omega_1 - \omega_2)}[S_+, \frac{1}{\hat{\hat{L}} - \mathrm{i}\omega_1}[S_-, \rho_{\mathrm{ss}}]]]$$
$$+ [S_+, \frac{1}{\hat{\hat{L}} - 2\mathrm{i}\omega_1}[S_-, \frac{1}{\hat{\hat{L}} - \mathrm{i}\omega_1}[S_-, \rho_{\mathrm{ss}}]]]\Big\}, \tag{12.44}$$

$\hat{\hat{L}} = \hat{\hat{L}}_r + \hat{\hat{L}}_c - \mathrm{i}\omega_0[\hat{S}_z,]$ and $\hat{\rho}_{\mathrm{ss}}$ is the steady state of $\hat{\hat{L}}$. For the sake of illustration, we asume that $\bar{N} = \bar{M} = 0$. In this case, $\hat{\rho}_{\mathrm{ss}} = |-1/2\rangle\langle -1/2|$. Verify also that

$$\hat{\hat{L}}|-1/2\rangle\langle 1/2| = [\mathrm{i}\omega_0 - (\gamma + \gamma_c)]\,|-1/2\rangle\langle 1/2|,$$
$$\hat{\hat{L}}|1/2\rangle\langle -1/2| = [-\mathrm{i}\omega_0 - (\gamma + \gamma_c)]\,|-1/2\rangle\langle 1/2|,$$
$$\hat{\hat{L}}\Big[|1/2\rangle\langle 1/2| - |-1/2\rangle\langle -1/2|\Big] = -2\gamma\Big[|1/2\rangle\langle 1/2|$$
$$-|-1/2\rangle\langle -1/2|\Big]. \tag{12.45}$$

It is now not difficult to show that

$$\chi^{(3)}(\omega_1, \omega_1, -\omega_2)$$
$$\sim \Big[-2(\gamma + \gamma_c) - \mathrm{i}(\omega_1 - \omega_2)\Big]\Big[-2\gamma - \mathrm{i}(\omega_1 - \omega_2)\Big]^{-1}$$
$$\times\Big[-\gamma - \gamma_c - \mathrm{i}(2\omega_1 - \omega_2 - \omega_0)\Big]^{-1}\Big[-\gamma - \gamma_c - \mathrm{i}(\omega_1 - \omega_0)\Big]^{-1}$$
$$\times\Big[-\gamma - \gamma_c + \mathrm{i}(\omega_2 - \omega_0)\Big]^{-1}. \tag{12.46}$$

The signal at $2\omega_1 - \omega_2$ as a function of ω_2 will, therefore, exhibit resonances at $\omega_0, 2\omega_1 - \omega_0$ and a resonance at ω_1 if $\gamma_c \neq 0$. The resonance at $\omega_2 = \omega_1$ is thus induced by collisions. It is referred to as the *Bloembergen resonance*. Note that the resonance at ω_1 does not correspond to an atomic transition. In Chap. 14 we will come across another important role played by collisions in four wave mixing in a cavity. It is in their ability to bring out otherwise suppressed resonances in non linear response of an atom in a cavity.

12.2 N Two-Level Atoms in a Monochromatic Field

The master equation for a system of N two-level atoms in contact with a squeezed reservoir is governed by (8.93). The Hilbert space of N two-level atoms, being a direct product of N two-dimensional spaces, is 2^N dimensional. Its density operator has, therefore, 2^{2N} elements. The system is evidently analytically intractable even for small $N > 1$ unless the atoms-reservoir interaction is such that it confines the evolution to some lower dimensional invariant

subspace. A case of common interest in which the evolution is confined to a space of drastically reduced dimension is that of the atoms occupying a volume so small that all of them see the same field. Such a situation is described by the master equation (8.95) if it is also assumed that $\Omega_{ij} \approx \nu$ for all i, j. For, then (8.95) would involve only collective two-level operators confining the evolution to a space of fixed total spin quantum number S. Now, for an N two-level system, $S = N/2, N/2 - 1, \ldots, 0$ or $1/2$ depending on whether N is even or odd. For a given S, the eigenvalues m of any component of the spin vector is such that $m = -S, -S+1, \ldots, S$. A possible state of two-level atoms is the one in which all the atoms are in the ground state. This corresponds to $m = -N/2$. This value of m is realized only if $S = N/2$. Hence, we restrict our attention to space of fixed $S = N/2$. Since the dimension of the space of spin quantum number S is $2S + 1$ it follows that the dimension of the density matrix obeying (8.95) is $(N+1)^2$. This is evidently an enormous reduction in the dimension compared with the general equation (8.93).

In the following we confine our attention to the collective operator form of (8.95). We let the system be driven by a monochromatic field assuming that the frequency Ω_p of the pump driving the squeezed bath is the same as the frequency ω_l of the field driving the atoms. The atomic density operator in the interaction picture defined by (12.3) then reads

$$\begin{aligned}\dot{\hat{\rho}}_{\mathrm{I}} = &-\mathrm{i}\left[\delta\hat{S}_z + g\hat{S}_+ + g^*\hat{S}_-,\ \hat{\rho}_{\mathrm{I}}\right] - \mathrm{i}\nu\left[\hat{S}_+\hat{S}_-,\ \hat{\rho}_{\mathrm{I}}\right] \\ &+\gamma\left[(\bar{N}+1)\left(2\hat{S}_-\hat{\rho}_{\mathrm{I}}\hat{S}_+ - \hat{\rho}_{\mathrm{I}}\hat{S}_+\hat{S}_- - \hat{S}_+\hat{S}_-\hat{\rho}_{\mathrm{I}}\right)\right. \\ &+\bar{N}\left(2\hat{S}_+\hat{\rho}_{\mathrm{I}}\hat{S}_- - \hat{\rho}_{\mathrm{I}}\hat{S}_-\hat{S}_+ - \hat{S}_-\hat{S}_+\hat{\rho}_{\mathrm{I}}\right) \\ &+\left\{\bar{M}\left(2\hat{S}_-\hat{\rho}_{\mathrm{I}}\hat{S}_- - \hat{\rho}_{\mathrm{I}}\hat{S}_-\hat{S}_- - \hat{S}_-\hat{S}_-\hat{\rho}_{\mathrm{I}}\right) + \mathrm{h.c.}\right\} \\ \equiv\ &\hat{\hat{L}}\hat{\rho}_{\mathrm{I}}. \end{aligned} \tag{12.47}$$

This is the master equation for the phenomenon of *collective resonance fluorescence* in a squeezed bath. We may solve this equation either by taking its matrix elements in the collective states $|S, m\rangle$ or by constructing the equations for averages. However, except for small values of N, these equations can be solved only numerically. We have already discussed the case of $N = 1$ in the last section. The analytical solution of (12.47) for $\nu = \bar{M} = 0$ for $N = 2$ and its numerical solution for N up to 20 are given respectively in [110] and [111].

Solving (12.47) even for its steady state is a non trivial task. For, it does not obey the condition (8.34) of detailed balance. Application of that condition requires us to construct the time-reversed form of the evolution operator. To do that, let us denote by $\hat{T}$ the operation of time reversal. Let $|\tilde{i}\rangle$ and $\hat{\tilde{A}}$ denote the state and the operator obtained by performing time-reversal on $|i\rangle$ and $\hat{A}$. The operator $\hat{T}$ is antilinear. We have, by definition, [112]

$$|\tilde{i}\rangle = \hat{T}|i\rangle, \qquad \tilde{\hat{A}} = \hat{T}^{-1}\hat{A}^{\dagger}\hat{T}. \tag{12.48}$$

If $\hat{A}$ in the equations above is a constant c then we find that $\tilde{c} = c^*$. We also note that (12.48) implies

$$\tilde{\hat{A}}\tilde{\hat{B}} = \tilde{\hat{B}}\tilde{\hat{A}}. \tag{12.49}$$

If $\hat{A} = \hat{S}_{\pm}$ then, invoking (12.48), we infer that

$$\tilde{\hat{S}}_{\pm} = \hat{S}_{\mp}, \qquad \tilde{\hat{S}}_z = \hat{S}_z. \tag{12.50}$$

We use these relations to find the time-reversed form of $\hat{\tilde{L}}$ given by (12.47). For the sake of illustration, we let $\delta = \bar{M} = \nu = 0$, $\bar{N} = \bar{n}$. We get

$$\begin{aligned}\hat{\tilde{L}}_{\rm tr}\hat{f} = {\rm i}\Big[&g\hat{S}_+ + g^*\hat{S}_-,\ \hat{f}\Big] \\ &+\gamma\Big[(\bar{n}+1)\Big(2\hat{S}_-\hat{f}\hat{S}_+ - \hat{f}\hat{S}_+\hat{S}_- - \hat{S}_+\hat{S}_-\hat{f}\Big) \\ &+\bar{n}\Big(2\hat{S}_+\hat{f}\hat{S}_- - \hat{f}\hat{S}_-\hat{S}_+ - \hat{S}_-\hat{S}_+\hat{f}\Big)\Big],\end{aligned} \tag{12.51}$$

The form of $\hat{\tilde{\tilde{L}}}$, found by invoking the definition (8.35), reads

$$\begin{aligned}\hat{\tilde{\tilde{L}}}\hat{f} = {\rm i}\Big[&g\hat{S}_+ + g^*\hat{S}_-,\ \hat{f}\Big] \\ &+\gamma\Big[(\bar{n}+1)\Big(2\hat{S}_+\hat{f}\hat{S}_- - \hat{f}\hat{S}_+\hat{S}_- - \hat{S}_+\hat{S}_-\hat{f}\Big) \\ &+\bar{n}\Big(2\hat{S}_-\hat{f}\hat{S}_+ - \hat{f}\hat{S}_-\hat{S}_+ - \hat{S}_-\hat{S}_+\hat{f}\Big).\end{aligned} \tag{12.52}$$

Substitute these relations in the detailed balance condition (8.34). It is straightforward to see that the resulting expression would hold for any $\hat{f}$ if

$$\begin{aligned}&\Big[g\hat{S}_+ + g^*\hat{S}_-,\ \hat{\rho}_{\rm ss}\Big] = 0, \\ &\Big[\hat{S}_+\hat{S}_-,\ \hat{\rho}_{\rm ss}\Big] = 0, \quad (\bar{n}+1)\hat{\rho}_{\rm ss}\hat{S}_+ - \bar{n}\hat{S}_+\hat{\rho}_{\rm ss} = 0.\end{aligned} \tag{12.53}$$

These equations can not be satisfied simultaneously if $g \neq 0$. However, if $g = 0$ then the last two equations above are solved by (3.102).

In the following we discuss the cases when analytical time-dependent and steady state solution of (12.47) can be derived. The time-dependent solutions can be found (a) if $g = \bar{M} = \bar{N} = 0$, (b) in the limit $\Omega \gg \gamma$, and (c) in the limit $N \gg 1$, whereas exact steady state solution is derivable analytically for (d) $m = \bar{n} = 0$, and (e) for some special values of the external field if $\nu = 0$, and $|\bar{M}|^2 = \bar{N}(\bar{N}+1)$.

$\boldsymbol{g = \bar{M} = \bar{N} = 0}$**.** The master equation for $g = \bar{M} = 0$ reads

$$\begin{aligned}\frac{{\rm d}\hat{\rho}}{{\rm d}t} = &-{\rm i}\nu\Big[\hat{S}_+\hat{S}_-,\ \hat{\rho}\Big] + \gamma(\bar{n}+1)\Big(2\hat{S}_-\hat{\rho}\hat{S}_+ - \hat{S}_+\hat{S}_-\hat{\rho} - \hat{\rho}\hat{S}_+\hat{S}_-\Big) \\ &+\gamma\bar{n}\Big(2\hat{S}_+\hat{\rho}\hat{S}_- - \hat{S}_-\hat{S}_+\hat{\rho} - \hat{\rho}\hat{S}_-\hat{S}_+\Big).\end{aligned} \tag{12.54}$$

It describes collective spontaneous emission in a thermal bath. It can be solved analytically if $\bar{n} = 0$. We may solve it by the method of eigenvectors expansion. To that end, take the matrix elements of (12.54) in the eigenstates $|S, m\rangle$ of $\hat{S}_z$. Define $\rho_{m+p,m} = \langle S-m-p|\hat{\rho}|S-m\rangle$, to arrive at the eigenvalue equation

$$\begin{aligned} &-\left[(\gamma - \mathrm{i}\nu)A_{m+1} + (\gamma + \mathrm{i}\nu)A_{m+p+1}\right]\rho_{m+p,m} \\ &+2\gamma\sqrt{A_m A_{m+p}}\rho_{m+p-1,m-1} = \lambda(p)\rho_{m+p,m}, \end{aligned} \tag{12.55}$$

$$A_m = m(N - m + 1). \tag{12.56}$$

This is to be solved as a recursion relation in m for a fixed p. The method of solving such an equation has been outlined in Sect. 10.3. We note that the eigenvalues are

$$\lambda_k(p) = (-\gamma + \mathrm{i}\nu)A_{k+1} + (-\gamma - \mathrm{i}\nu)A_{k+p+1}, \tag{12.57}$$

$k = 0, 1, \ldots, N$. The real part of the eigenvalues is non positive. They are non-degenerate for any $p \neq 0$ if $\nu \neq 0$. The eigenvalues are two-fold degenerate if $p = 0$. For, then

$$\lambda_k(0) = -2\gamma(k+1)(N-k) = \lambda_{N-k-1}(0), \quad k = 0, 1, \ldots, N. \tag{12.58}$$

The eigenvalue 0 corresponding to $k = N$ is clearly non degenerate. The solution of (12.55) in case of $p = 0$ is to be found by constructing its generalized eigenvectors. It is a tedious but straightforward task. The reader may carry it for $N = 2$ and compare the solution with the one derived by direct integration of the time dependent equation [89]. We have outlined that method in Sect. 10.3.

The High-Field Limit. Recall from Sect. 8.2 that the dominant contribution to a master equation in this limit may be extracted by applying the secular approximation. The task at hand is to solve (12.47) in the limit when the hamiltonian part in it is strong compared with the damping. This task is accomplished conveniently by transforming to a new set of spin operators defined by (assumimg g to be read)

$$\begin{pmatrix} \hat{R}_x \\ \hat{R}_y \\ \hat{R}_z \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -\delta/\Omega & 0 & 2g/\Omega \\ 2g/\Omega & 0 & \delta/\Omega \end{pmatrix} \begin{pmatrix} \hat{S}_x \\ \hat{S}_y \\ \hat{S}_z \end{pmatrix}, \tag{12.59a}$$

$$\begin{pmatrix} \hat{S}_x \\ \hat{S}_y \\ \hat{S}_z \end{pmatrix} = \begin{pmatrix} 0 & -\delta/\Omega & 2g/\Omega \\ 1 & 0 & 0 \\ 0 & 2g/\Omega & \delta/\Omega \end{pmatrix} \begin{pmatrix} \hat{R}_x \\ \hat{R}_y \\ \hat{R}_z \end{pmatrix}, \tag{12.59b}$$

the Ω being the Rabi frequency defined in (12.34). The equation (12.47) then reads

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = -\mathrm{i}\Omega\left[\hat{R}_z,\ \hat{\rho}\right] + \hat{\hat{L}}\hat{\rho}. \tag{12.60}$$

The superoperator $\hat{\hat{L}}$ is written, of course, in terms of the new set of operators defined in (12.59a). Now, define

$$\hat{\rho}_{\mathrm{I}}(t) = \exp\left(\mathrm{i}\Omega\hat{R}_z t\right)\hat{\rho}(t)\exp\left(-\mathrm{i}\Omega\hat{R}_z t\right), \tag{12.61}$$

so that $\hat{\rho}_{\mathrm{I}}(t)$ obeys the equation

$$\frac{\mathrm{d}\hat{\rho}_{\mathrm{I}}}{\mathrm{d}t} = \hat{\hat{L}}_I(t)\hat{\rho}_{\mathrm{I}}(t), \tag{12.62}$$

where

$$\hat{\hat{L}}_I(t) = \exp\left(\mathrm{i}\Omega\hat{R}_z t\right)\hat{\hat{L}}\exp\left(-\mathrm{i}\Omega\hat{R}_z t\right). \tag{12.63}$$

The $\hat{\hat{L}}_I(t)$ would contain terms some of which are independent of time and the terms which oscillate at the Rabi frequency or its multiples. As discussed in Sect. 8.3, if the hamiltonian part is very much strong compared with damping, i.e. if $\gamma \ll \Omega$ then the dominant contribution to the master equation arises from time-independent terms in $\hat{\hat{L}}_I$. This constitutes the secular approximation. It is straightforward to see that only the terms in which $\hat{\rho}$ occurs in combination with equal number of $\hat{R}_+$ and $\hat{R}_-$, and any power of $\hat{R}_z$ are time-independent under the transformation generated by $\hat{R}_z$. Collecting all such terms together we find that, in the secular approximation, (12.60) reduces to

$$\begin{aligned}\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = &-\mathrm{i}\Omega\left[\hat{R}_z,\ \hat{\rho}\right] - \mathrm{i}\nu\left[\Delta_+^2\hat{R}_+\hat{R}_- + \Delta_-^2\hat{R}_-\hat{R}_+ + \frac{4g^2}{\Omega^2}\hat{R}_z^2,\ \hat{\rho}\right]\\ &+\Gamma_1\left[2\hat{R}_-\rho\hat{R}_+ - \rho\hat{R}_+\hat{R}_- - \hat{R}_+\hat{R}_-\rho\right]\\ &+\Gamma_2\left[2\hat{R}_+\rho\hat{R}_- - \rho\hat{R}_-\hat{R}_+ - \hat{R}_-\hat{R}_+\rho\right]\\ &+\Delta_z\left[2\hat{R}_z\rho\hat{R}_z - \rho\hat{R}_z\hat{R}_z - \hat{R}_z\hat{R}_z\rho\right],\end{aligned} \tag{12.64}$$

$$\begin{aligned}\Delta_\pm &= \frac{1}{2}\left(1 \pm \frac{\delta}{\Omega}\right), \qquad \Delta_z = \gamma\frac{4g^2}{\Omega^2}(2\bar{n}+1),\\ \Gamma_1 &= \gamma\left\{(\bar{n}+1)\Delta_+^2 + \bar{n}\Delta_-^2\right\},\\ \Gamma_2 &= \gamma\left\{(\bar{n}+1)\Delta_-^2 + \bar{n}\Delta_+^2\right\}.\end{aligned} \tag{12.65}$$

Verify that the steady state solution of this equation is

$$\hat{\rho}_{\mathrm{ss}} = \exp\left(-\ln(\Gamma_1/\Gamma_2)\hat{R}_z\right)\Big/\mathrm{Tr}\left[\exp\left(-\ln(\Gamma_1/\Gamma_2)\hat{R}_z\right)\right]. \tag{12.66}$$

For the sake of illustration, we write the equations for averages of operators corresponding to (12.64). If $\nu = 0$ then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\langle\hat{R}_+\right\rangle = &\ \mathrm{i}\Omega\left\langle\hat{R}_+\right\rangle + 2\Gamma_1\left\langle\hat{R}_+\hat{R}_z\right\rangle - 2\Gamma_2\left\langle\hat{R}_z\hat{R}_+\right\rangle\\ &-\Delta_z\left\langle\hat{R}_+\right\rangle\end{aligned}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \hat{R}_z \right\rangle = -2\Gamma_1 \left\langle \hat{R}_+ \hat{R}_- \right\rangle + 2\Gamma_2 \left\langle \hat{R}_- \hat{R}_+ \right\rangle . \tag{12.67}$$

The equation for an operator average is coupled with the average of the products of operators. Each of these products is reduceuble to a single operator in the case of a single atom system. We leave it to the reader to solve these equations for $N = 1$. In general, we note that if $\Gamma_1 = \Gamma_2$ then the terms containing products of operators combine to reduce to commutators. As a result, each of the equation above is closed. Note that we can have $\Gamma_1 = \Gamma_2$ provided $\delta = 0$. We also let $\bar{n} = 0$ so that $\Gamma_1 = \Gamma_2 = \gamma/4$ and $\Delta_z = \gamma$. The solution of (12.67) then reads

$$\begin{aligned} \left\langle \hat{R}_+(t) \right\rangle &= \exp\left(2\mathrm{i}gt - \frac{3\gamma t}{2}\right) \left\langle \hat{R}_+ \right\rangle , \\ \left\langle \hat{R}_z(t) \right\rangle &= \exp\left(-\gamma t\right) \left\langle \hat{R}_z \right\rangle . \end{aligned} \tag{12.68}$$

The steady state in this case is

$$\hat{\rho}_{\mathrm{ss}} = I/(N+1). \tag{12.69}$$

We leave it to the reader to show that

$$G(\tau) = \frac{N(N+2)}{24}\Big[2\exp(-\gamma t) + \{\exp(2\mathrm{i}gt - 3\gamma t/2) + \mathrm{c.c.}\}\Big]. \tag{12.70}$$

From this we infer that the emission spectrum consists of three Lorentzians, one centred at the driving field frequency and the other two displaced by $\pm 2g$ from it. The central peak has the width γ whereas the width of each of the other two peaks is $3\gamma/2$.

***N* ≫ 1 .** In what follows, it will be found to be useful to work with scaled parameters defined by the relations

$$\tau = \gamma N t, \quad \theta = \frac{2g}{\gamma N}, \quad \Delta = \frac{\delta}{\gamma N}, \quad \tilde{\nu} = \frac{\nu}{\gamma}. \tag{12.71}$$

It will be assumed that $N\gamma$ is finite in the limit $N \gg 1$.

The theory of the behaviour of the atomic system in this limit is based on expressing the averages and the variances as

$$\begin{aligned} \frac{\left\langle \hat{S}_\mu \right\rangle}{N} &= m_\mu^{(0)} + \epsilon m_\mu^{(1)} + \cdots , \\ \frac{\left\langle \hat{S}_\mu \hat{S}_\nu \right\rangle - \left\langle \hat{S}_\mu \right\rangle \left\langle \hat{S}_\nu \right\rangle}{N^2} &= \epsilon \sigma_{\mu\nu}^{(1)} + \epsilon^2 \sigma_{\mu\nu}^{(2)} + \cdots , \quad \epsilon = \frac{1}{N}. \end{aligned} \tag{12.72}$$

and so on. Now , with $\bar{N} = \bar{M} = 0$ and the parameters scaled as in (12.71), (12.47) yields

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\left\langle \hat{S}_+ \right\rangle}{N} &= \mathrm{i}\Delta \frac{\left\langle \hat{S}_+ \right\rangle}{N} - 2\mathrm{i}\tilde{\nu}\frac{\left\langle \hat{S}_+ \hat{S}_z \right\rangle}{N^2} - \mathrm{i}\theta \frac{\left\langle \hat{S}_z \right\rangle}{N} + 2\frac{\left\langle \hat{S}_+ \hat{S}_z \right\rangle}{N^2}, \\ \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\left\langle \hat{S}_z \right\rangle}{N} &= -\frac{\mathrm{i}\theta}{2}\frac{\left\langle \hat{S}_+ - \hat{S}_- \right\rangle}{N} - 2\frac{\left\langle \hat{S}_+ \hat{S}_- \right\rangle}{N^2}. \end{aligned} \tag{12.73}$$

On using (12.72) and on retaining the terms to zeroth order in ϵ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\tau}m_+^{(0)} = \mathrm{i}\Delta m_+^{(0)} + \left\{2m_+^{(0)}(1-\mathrm{i}\tilde{\nu}) - \mathrm{i}\theta\right\} m_z^{(0)},$$
$$\frac{\mathrm{d}}{\mathrm{d}\tau}m_z^{(0)} = -2m_+^{(0)}m_-^{(0)} - \frac{\mathrm{i}\theta}{2}\left(m_+^{(0)} - m_-^{(0)}\right). \tag{12.74}$$

To the same order in ϵ, $\langle\hat{S}^2\rangle/N^2 = 1/4$ which, on using the relation $\hat{S}^2 = \hat{S}_+\hat{S}_- + \hat{S}_z^2 - \hat{S}_z$ gives

$$m_+^{(0)}m_-^{(0)} + m_z^{(0)2} = \frac{1}{4}. \tag{12.75}$$

By applying (12.74), it can be checked that the time derivative of this expression is indeed zero. As an illustration, we solve (12.74) for its steady state by equating the time derivatives to zero. Show that the inversion in the steady state solves the quartic

$$\left[\left(\Delta - 2m_z^{(0)}\tilde{\nu}\right)^2 + 4m_z^{(0)2}\right]\left[\frac{1}{4} - m_z^{(0)2}\right] - \theta^2 m_z^{(0)2} = 0. \tag{12.76}$$

Its solution for $\nu = \delta = 0$ yields

$$m_z^{(0)} = \pm\frac{1}{2}\sqrt{1-\theta^2}, \qquad m_+^{(0)} = \frac{\mathrm{i}\theta}{2}, \qquad \theta \leq 1, \tag{12.77a}$$

$$m_z^{(0)} = 0, \qquad m_+^{(0)} = \frac{1}{2\theta}\sqrt{\theta^2-1} \pm \frac{\mathrm{i}}{2\theta} \quad \theta > 1. \tag{12.77b}$$

These solutions must be examined for their stability by determining how a small disturbance of the steady state develops in time. To that end, we let $m_\mu = m_\mu^{(0)} + \delta m_\mu$ in (12.74) and derive the equations obeyed by δm_μ. We find that ($\Delta = \nu = 0$)

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{pmatrix}\delta m_z \\ \delta m_+ - \delta m_-\end{pmatrix} = \begin{pmatrix}4m_z^0 & -\mathrm{i}\theta/2 \\ 2\{m_+^{(0)} - m_-^{(0)} - \mathrm{i}\theta\} & 2m_z^0\end{pmatrix} \times \begin{pmatrix}\delta m_z \\ \delta m_+ - \delta m_-\end{pmatrix} \tag{12.78}$$

The nature of evolution is determined by the eigenvalues of the matrix of evolution. It is straightforward to see that if $\theta \leq 1$ then the eigenvalues of the matrix of evolution in (12.78) are $2m_z^0, 4m_z^0$. The steady state is stable if the real part of all the eigenvalues is negative. Hence, (12.77a) is a stable solution if $m_z^0 < 0$. The eigenvalues of the evolution matrix for $\theta \geq 1$ are $\pm\mathrm{i}\sqrt{\theta^2-1}$. The significance of purely imaginary eigenvalues is revealed by comparing the approximate results with exact asymptotic results given in (12.91) and (12.92). We note that the approximate expressions for the averages above are in agreement with the exact expressions for $\theta \leq 1$. In this range, the eigenvalues of the matrix of evolution have a negative real part. However, the approximate results for $\theta > 1$ are wide off the exact results. Thus fluctuations, ignored in obtaining the approximate results in (12.77b),

seem to play important role in driving the system towards the steady state above $\theta = 1$.

The equations (12.74) can be solved exactly for $\delta = \nu = 0$. Below $\theta = 1$, the time-dependence is purely decaying. Above $\theta = 1$, the system follows a closed trajectory around the steady state solution determined by the initial condition. But for ignored fluctuations, the system would stay on a trajectory. These fluctuations influence the motion on a time scale much longer than its period. They cause the motion to diffuse on a trajectory as well as between the trajectories. The dynamics of the system on that time scale may be viewed as a result of averaging over the motion on all the trajectories and the initial states. A systematic procedure for carrying such an average is outlined in [113].

Observe that the approximate as well as the exact steady state results show that, for $\delta = \nu = 0$, the steady state averages are continuous functions of the driving field parameter θ but their derivatives with respect to that parameter are discontinuous at $\theta = 1$. This behaviour is reminiscent of a second-order phase transition.

The nature of the steady state is entirely different if $\nu,\ \delta \neq 0$. In this case (12.76) may be solved numerically. Of course, only its stable real solutions in the range $[-1/2, 1/2]$ are physically acceptable. In order to bring out essential features of the solution, we have plotted in Fig. 12.1 $m_z^{(0)}$ as a function of θ for $\Delta = 0.5,\ \tilde{\nu} = -5.0$. These are the values of the parameters used in [114].

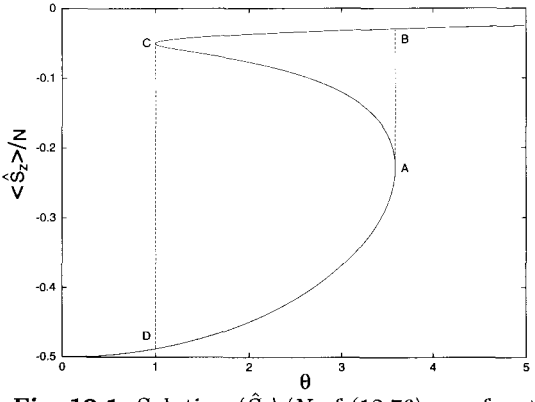

Fig. 12.1. Solution $\langle \hat{S}_z \rangle / N$ of (12.76) as a function of θ for $\Delta = .5,\ \tilde{\nu} = -5.0$.

We have omitted the part of the curve corresponding to positive values of $m_z^{(0)}$ as that part is unstable (see [114]). The part of the curve between points A and C is also unstable. Let θ_A and θ_C be the values of θ corresponding to the points A and C respectively. Notice that, for the values of θ in the range (θ_C, θ_A), the system admits two stable steady states whereas there is one stable state outside the said range. In order to see which of the two

states the system exists in, let θ increase from $\theta = 0$. The system would follow the lower curve up to the the point A corresponding to $\theta = \theta_A$. As the field is increased beyond the value corresponding to $\theta = \theta_A$, the steady state of the system would jump on to B and follow the upper curve. This is reminiscent of the first order phase transition. If the field is decreased while the system is on the upper branch, the system would follow it till the point C corresponding to $\theta = \theta_C$ where it will jump to D and follow the lower branch. Thus, transition from the upper to the lower branch takes place at a value of θ that is different from the value at which it jumps from the lower to the upper branch. This exemplifies the phenomenon of *hysteresis*. We will see that whereas the exact steady state solution derived below does predict first-order phase transition, it does not predict hysteresis. The hysteresis is thus an artifact of the decorrelation which ignores quantum fluctuations.

The Steady State for $\bar{M} = \bar{N} = 0$. The master equation in this case reads

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = -\mathrm{i}g\left[\hat{S}_+ + \hat{S}_-,\ \hat{\rho}\right] - \mathrm{i}\delta\left[\hat{S}_z,\ \hat{\rho}\right] - \mathrm{i}\nu\left[\hat{S}_+\hat{S}_-,\ \hat{\rho}\right] + \gamma\left(2\hat{S}_-\hat{\rho}\hat{S}_+ - \hat{S}_+\hat{S}_-\hat{\rho} - \hat{\rho}\hat{S}_+\hat{S}_-\right). \tag{12.79}$$

The steady state solution of this equation can be readily derived if $\delta = 0$ by rewriting it in the easily verifiable form

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = 2\gamma\left(\hat{S}_- + \mathrm{i}G\right)\hat{\rho}\left(\hat{S}_+ - \mathrm{i}G^*\right) - (\gamma + \mathrm{i}\nu)\left(\hat{S}_+ - \mathrm{i}G^*\right)\left(\hat{S}_- + \mathrm{i}G\right)\hat{\rho} - (\gamma - \mathrm{i}\nu)\,\hat{\rho}\left(\hat{S}_+ - \mathrm{i}G^*\right)\left(\hat{S}_- + \mathrm{i}G\right), \tag{12.80}$$

where

$$G = \frac{g}{\gamma + \mathrm{i}\nu}. \tag{12.81}$$

It is straightforward to see that

$$\hat{\rho}_{\mathrm{ss}} \sim \left(\hat{S}_- + \mathrm{i}G\right)^{-1}\left(\hat{S}_+ - \mathrm{i}G^*\right)^{-1} = \frac{1}{D}\sum_{m,n=0}^{N}(\mathrm{i}G)^{-m}(-\mathrm{i}G^*)^{-n}\hat{S}_-^m\hat{S}_+^n \tag{12.82}$$

is the steady state solution of (12.80). Note that the inverse operators in the first line above exist and that the upper limit in the second line is restricted to N owing to the fact that $\hat{S}_\pm^{N+m} = 0$ for $m \geq 1$. The constant D is chosen to have $\mathrm{Tr}(\hat{\rho}_{\mathrm{ss}}) = 1$. The solution for $\nu = 0$ was derived first in [115]. In the absence of an external drive, i.e. for $g = 0$, (12.82) reduces to

$$\hat{\rho}_{\mathrm{ss}} = \frac{\hat{S}_-^N\hat{S}_+^N}{\mathbf{Tr}\left[\hat{S}_-^m\hat{S}_+^n\right]} \equiv |-N/2\rangle\langle -N/2|. \tag{12.83}$$

It correctly shows that, in the absence of a drive, the atoms at absolute zero temperature settle to their ground state.

Guided by the expression (12.82) for the steady state for $\delta = 0$, we let the steady state of (12.79) for $\delta \neq 0$ to be of the form

$$\hat{\rho}_{\rm ss} = \sum_{m,n=0}^{N} C_{mn} \hat{S}_-^m S_+^n. \tag{12.84}$$

Substitute this in the steady state version of (12.79) and write the detuning term as [117]

$$\begin{aligned} -\mathrm{i}\delta \left[\hat{S}_z,\ \hat{S}_-^m \hat{S}_+^n\right] &= \mathrm{i}\delta \left[\hat{S}_z,\ \hat{S}_-^m \hat{S}_+^n\right] - 2\mathrm{i}\delta \left[\hat{S}_z,\ \hat{S}_-^m \hat{S}_+^n\right] \\ &= \mathrm{i}\delta(n-m)\ \hat{S}_-^m \hat{S}_+^n - 2\mathrm{i}\delta \left[\hat{S}_z \hat{S}_-^m \hat{S}_+^n - \hat{S}_-^m \hat{S}_+^n \hat{S}_z\right]. \end{aligned} \tag{12.85}$$

Verify also that

$$\begin{aligned} &\left[\hat{S}_+ \hat{S}_-,\ \hat{S}_-^m \hat{S}_+^n\right] \\ &= (m+1)(2\hat{S}_z + m)\hat{S}_-^m \hat{S}_+^n - (n+1)\hat{S}_-^m \hat{S}_+^n (2\hat{S}_z + n). \end{aligned} \tag{12.86}$$

The damping part may be treated similarly to arrive at

$$\begin{aligned} &\sum_{m,n=0}^{N} \Bigg[\left\{ \Big((m+1)(\gamma + \mathrm{i}\nu) + \mathrm{i}\delta \Big) C_{mn} + \mathrm{i}g(m+1) C_{m+1n} \right\} \\ &\times \left\{ m\hat{S}_-^m \hat{S}_+^n + 2\hat{S}_z \hat{S}_-^m \hat{S}_+^n \right\} + \text{h.c.} \Bigg] = 0 \end{aligned} \tag{12.87}$$

It is readily seen that this equation is satisfied if

$$\left(m + 1 + \frac{\mathrm{i}\delta}{\gamma + \mathrm{i}\nu} \right) C_{mn} + \frac{\mathrm{i}g(m+1)}{\gamma + \mathrm{i}\nu} C_{m+1n} = 0. \tag{12.88}$$

This is solved by

$$C_{mn} \sim \frac{1}{m!n!} (\mathrm{i}G)^{-m} (-\mathrm{i}G^*)^{-n} \Gamma\left(m + 1 - \mathrm{i}\phi\right) \Gamma\left(n + 1 + \mathrm{i}\phi^*\right), \tag{12.89}$$

$$\phi = \frac{-\mathrm{i}\delta}{\gamma + \mathrm{i}\nu}. \tag{12.90}$$

The exact steady state solution for $\nu, \delta \neq 0$ was derived in [114]. The steady state for $\nu = 0, \delta \neq 0$ is derived in [117]. The averages may be evaluated by using (3.71). For the sake of comparison with the results derived above in the limit $N \gg 1$, we express the averages in terms of parameters normalized as in (12.71). We exhibit the behaviour of $\langle \hat{S}_z \rangle / N$ as a function of θ in Fig. 12.2 for $\Delta = .5$, $\bar{\nu} = -5.0$ corresponding to $N = 10, 30, 60$. Shown along with it is also the curve of Fig. 12.1 depicting the behaviour in the decorrelation approximation expected to be valid in the limit $N \gg 1$. The exact

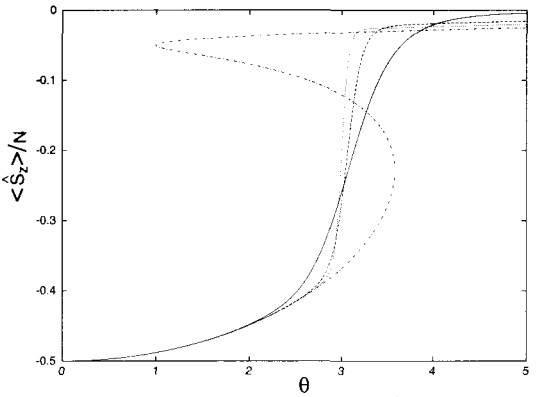

Fig. 12.2. Steady state value of $\langle \hat{S}_z \rangle / N$ as a function of θ for $\Delta = .5, \tilde{\nu} = -5.0$ and for $N = 10$ (solid line), $N = 30$ (short dashes), $N = 60$ (long dashes). Reproduced also is the plot of Fig.12.1 (dashed-dotted curve) depicting the results of the decorrelation approximation.

average exhibits a tendency towards first-order phase transition in agreement with the decorrelation approximation. However, exact result does not predict hysteresis.

The expressions for averages assume simpler form in the asymptotic limit $N \gg 1$ if $\Omega = \delta = 0$. It has been shown that in this case, [115]

$$\left\langle \hat{S}_x \right\rangle = 0, \quad \left\langle \hat{S}_y \right\rangle / N = \frac{\theta}{2}, \quad \left\langle \hat{S}_z \right\rangle / N = -\frac{1}{2}\sqrt{1-\theta^2},$$
$$\sigma_{\mu\mu} = \left(\left\langle \hat{S}_\mu^2 \right\rangle - \left\langle \hat{S}_\mu \right\rangle^2 \right) / N^2 = 0, \quad \theta \leq 1, \tag{12.91}$$

and

$$\left\langle \hat{S}_z \right\rangle / N = 0, \quad \left\langle \hat{S}_y \right\rangle / N = \frac{\theta}{2}\left[1 - \frac{\sqrt{\theta^2-1}}{\theta^2 \sin^{-1}(1/\theta)} \right],$$
$$\sigma_{xx} = \frac{\sqrt{\theta^2-1}}{12\theta^2 \sin^{-1}(1/\theta)},$$
$$\sigma_{yy} = \frac{\sqrt{\theta^2-1}}{4\sin^{-1}(1/\theta)}\left[1 - \frac{1}{3\theta^2} - \frac{\sqrt{\theta^2-1}}{\theta^2 \sin^{-1}(1/\theta)} \right],$$
$$\sigma_{zz} = \frac{1}{4} - \frac{\theta^2}{4}\left[1 - \frac{\sqrt{\theta^2-1}}{\theta^2 \sin^{-1}(1/\theta)} \right], \quad \theta \geq 1. \tag{12.92}$$

This shows that the zero-variance approximation is valid if $\theta \leq 1$ but not if $\theta > 1$. However, the results found under the zero variance approximation, along with averaging for $\theta \geq 1$ mentioned above, are in agreement with exact asymptotic results [113]. We display in Fig. 12.3 the behaviour of $\langle \hat{S}_z \rangle / N$ as a function of θ for $N = 10, 30, 60$ and the asymptotic result derived above.

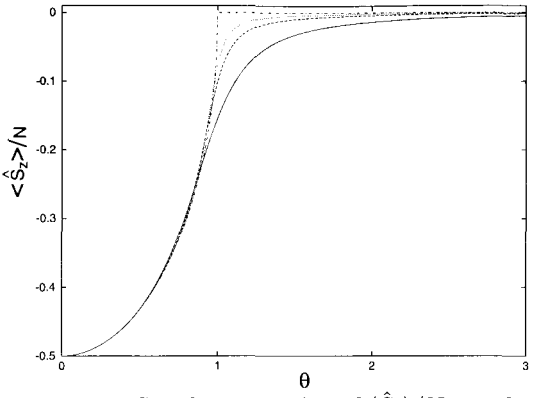

Fig. 12.3. Steady state value of $\langle \hat{S}_z \rangle / N$ as a function of θ. The curves, in order of the lowermost to the uppermost, correspond to $N = 10, 30, 60$ and the asymptotic result $N \gg 1$.

The plots for finite N exhibit the expected tendency of approaching the asymptotic one with an increase in the value of N.

Steady State for $\delta = g = \nu = 0$, $\bar{M} \neq 0$. Finding the steady state of (12.47) in this case is facilitated by rewriting it as

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho} = \gamma(\bar{n}+1)\left(2\hat{R}_z\hat{\rho}\hat{R}_z^\dagger - \hat{R}_z^\dagger\hat{R}_z\hat{\rho} - \hat{\rho}\hat{R}_z^\dagger\hat{R}_z\right) + \gamma\bar{n}\left(2\hat{R}_z^\dagger\hat{\rho}\hat{R}_z - \hat{R}_z\hat{R}_z^\dagger\hat{\rho} - \hat{\rho}\hat{R}_z R_z^\dagger\right). \tag{12.93}$$

where

$$\begin{aligned} \bar{M} &= \mu\nu(1+2\bar{n})\exp(\mathrm{i}\phi), \\ \bar{N} &= |\mu|^2\bar{n} + (1+\bar{n})|\nu|^2, \quad |\mu|^2 - |\nu|^2 = 1, \end{aligned} \tag{12.94}$$

$$\hat{R}_z = \left[|\mu|\exp(\mathrm{i}\phi)\hat{S}_- + |\nu|\hat{S}_+\right]. \tag{12.95}$$

It can be solved analytically for its steady state if $\bar{n} = 0$. Recall from Chap. 3 that the eigenvalues of $\hat{R}_z$ and hence those of $\hat{R}_z^\dagger$ are $m = -S, -S+1, \cdots, S$ with $S = N/2$. This implies that zero is an eigenvalue of $\hat{R}_z$ and $\hat{R}_z^\dagger$ if N is even. The steady state of (12.93) for $\bar{n} = 0$ then is

$$\hat{\rho}_{\mathrm{ss}} = |\psi_0\rangle\langle\psi_0|, \quad \hat{R}_z|\psi_0\rangle = 0. \tag{12.96}$$

Since $\hat{R}_z$ is a linear combination of $\hat{S}_x$ and $\hat{S}_y$, $|\psi_0\rangle$ is their minimum uncertainty state. We have exhibited the behaviour of $|\langle S, m|\psi_0\rangle|^2$ in Fig. 3.2 as a function of m. See also the discussion following that figure.

However, if N is odd then zero is no longer an eigenvalue. But then $\hat{R}_z^{-1}$ exists. As a consequence

$$\hat{\rho}_{\mathrm{ss}} \sim \hat{R}_z^{-1}\hat{R}_z^{-1\dagger}, \tag{12.97}$$

is the steady state of (12.93) for $\bar{n} = 0$. For further details of its properties and numerical results, see [48].

The steady state of the master equation for $\delta \neq 0$ and for some special values of the driving field has been derived in [48]. Those solutions exhibit several interesting quantum features.

12.3 Two-Level Atoms in a Fluctuating Field

So far we have treated the field as a deterministic function of time. Now, we let the field to be fluctuating and assume that it is of the form

$$E(t) = \exp(-\mathrm{i}\phi(t) - \mathrm{i}\omega_l t)\,(E_0 + \epsilon(t)) + \mathrm{h.c.}. \tag{12.98}$$

where the phase $\phi(t)$ and $\epsilon(t)$ are fluctuating variables, and E_0 is a constant. The problem is analytically manageable if the reservoir is thermal, i.e. if $\bar{M} = 0$. We assume that to be the case and rewrite the master equation (12.47) in the form

$$\dot{\hat{\rho}}_{\mathrm{I}} = -\mathrm{i}\Big[(g_0 + g(t))\exp(-\mathrm{i}\phi(t))\hat{S}_+ + \mathrm{h.c.}, \hat{\rho}_{\mathrm{I}}\Big] + \hat{\hat{L}}\hat{\rho}_{\mathrm{I}}. \tag{12.99}$$

The $\hat{\hat{L}}$ in the equation above is time-independent. Define

$$\hat{W}_m = \exp\left(-\mathrm{i}m\phi(t)\right)\exp\left(\mathrm{i}\phi(t)\hat{S}_z\right)\hat{\rho}_{\mathrm{I}}(t)\exp\left(-\mathrm{i}\phi(t)\hat{S}_z\right). \tag{12.100}$$

The $\hat{W}_m$ averaged over the distribution of $\phi(t)$, denoted by $\overline{\hat{W}}_m$, provides information about the expectation value of the operators averaged over the distribution of $\phi(t)$. For,

$$\begin{aligned}
\overline{\langle \hat{S}_+^p \hat{S}_z^q \hat{S}_-^{p+m}\rangle} &= \mathrm{Tr}\left[\overline{\hat{\rho}_{\mathrm{I}}(t)\hat{S}_+^p \hat{S}_z^q \hat{S}_-^{p+m}}\right] \\
&= \mathrm{Tr}\left[\overline{\exp\left(+\mathrm{i}m\phi(t)\right)\exp\left(-\mathrm{i}\phi(t)\hat{S}_z\right)\hat{W}_m\exp\left(\mathrm{i}\phi(t)\hat{S}_z\right)\hat{S}_+^p\hat{S}_z^q\hat{S}_-^{p+m}}\right] \\
&= \mathrm{Tr}\left[\overline{\hat{W}}_m\hat{S}_+^p\hat{S}_z^q\hat{S}_-^{p+m}\right].
\end{aligned} \tag{12.101}$$

Hence, any expectation value averaged over the phase fluctuations is determined by an appropriately chosen $\overline{\hat{W}}_m$. For example, the expectation value of an operator that is diagonal in the eigenstates of $\hat{S}_z$ is determined by $\overline{\hat{W}}_0$ and that the expectation value of $\hat{S}_\pm$, needed for evaluating the dipole moment, is governed by $\overline{\hat{W}}_{\mp 1}$. On substituting (12.100) in (12.99) we find that W_m obeys the equation

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\hat{W}_{\mathrm{m}} = &-\mathrm{i}\phi\Big[m\hat{W}_{\mathrm{m}} - [\hat{S}_z, \hat{W}_{\mathrm{m}}]\Big] + \Big[(g_0 + g(t))\hat{S}_+ + \mathrm{h.c.}, \hat{W}_{\mathrm{m}}\Big] \\
&+\hat{\hat{L}}\hat{W}_{\mathrm{m}},
\end{aligned} \tag{12.102}$$

The equation for $\hat{W}_{\rm m}$ averaged over the fluctuations of $\phi(t)$ can be derived if $\dot{\phi}(t)$ is a delta-correlated Gaussian process. Let

$$\overline{\dot{\phi}(t)} = 0, \quad \overline{\dot{\phi}(t)\dot{\phi}(t')} = 2\gamma_p \delta(t-t'). \tag{12.103}$$

Following Chap. 5 we find that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\overline{\hat{W}}_{\rm m} = \gamma_p\Big[2\hat{S}_z\overline{\hat{W}}_{\rm m}\hat{S}_z - \hat{S}_z^2\overline{\hat{W}}_{\rm m} - \overline{\hat{W}}_{\rm m}\hat{S}_z^2\Big] - m\gamma_p\Big(m\overline{\hat{W}}_{\rm m} - 2[\hat{S}_z, \overline{\hat{W}}_{\rm m}]\Big) \\ -\mathrm{i}\Big[(g_0+g(t))\hat{S}_+ + \mathrm{h.c.}, \overline{\hat{W}}_{\rm m}\Big] + \hat{\hat{L}}\overline{\hat{W}}_{\rm m}.\end{aligned} \tag{12.104}$$

This may be solved analytically if $g(t)$ is a constant. See [116] for details and the numerical results.

The case of coloured $\dot{\phi}(t)$ can be handled by treating the equation for $\hat{W}$ by the method outlined in Chap. 5. For some numerical results, see [70].

12.4 Driven Three-Level Atom

Consider a three-level atom in the ladder configuration interacting with the field reservoir. Its evolution is described by the master equation (8.109). For the sake of simplicity, we let the reservoir to be thermal at absolute zero temperature. The evolution of the atom is then governed by (8.109) with $\bar{N} = \bar{M} = 0$ reading

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho} = -\mathrm{i}\left[\sum_{j=\mathrm{g,i,e}} E_j\hat{A}_{jj}, \hat{\rho}\right] + 2\gamma_{\rm x}\left[\hat{A}_{\rm ie}\hat{\rho}\hat{A}_{\rm ig} + \hat{A}_{\rm gi}\hat{\rho}\hat{A}_{\rm ei}\right] \\ +\gamma_1\left[2\hat{A}_{\rm gi}\hat{\rho}\hat{A}_{\rm ig} - \rho\hat{A}_{\rm ii} - \hat{A}_{\rm ii}\rho\right] \\ +\gamma_2\left[2\hat{A}_{\rm ie}\hat{\rho}\hat{A}_{\rm ei} - \rho\hat{A}_{\rm ee} - \hat{A}_{\rm ee}\rho\right].\end{aligned} \tag{12.105}$$

It should be borne in mind that the term multiplying $\gamma_{\rm x}$ contributes as long as $E_{\rm i} - E_{\rm g} \sim E_{\rm e} - E_{\rm i}$ (see Chap. 8). The equations for the operator averages read

$$\begin{aligned}\langle\hat{\hat{A}}_{\rm gi}\rangle &= -\mathrm{i}(E_{\rm i}-E_{\rm g})\langle\hat{A}_{\rm gi}\rangle - \gamma_1\langle\hat{A}_{\rm gi}\rangle + 2\gamma_{\rm x}\langle\hat{A}_{\rm ie}\rangle \\ \langle\hat{\hat{A}}_{\rm ie}\rangle &= -\mathrm{i}(E_{\rm e}-E_{\rm i})\langle\hat{A}_{\rm ie}\rangle - (\gamma_1+\gamma_2)\langle\hat{A}_{\rm ie}\rangle, \\ \langle\hat{\hat{A}}_{\rm ge}\rangle &= -\mathrm{i}(E_{\rm e}-E_{\rm g})\langle\hat{A}_{\rm ge}\rangle - \gamma_2\langle\hat{\hat{A}}_{\rm ge}\rangle \\ \langle\hat{\hat{A}}_{\rm gg}\rangle &= 2\gamma_1\langle\hat{A}_{\rm ii}\rangle, \qquad \langle\hat{\hat{A}}_{\rm ee}\rangle = -2\gamma_2\langle\hat{A}_{\rm ee}\rangle.\end{aligned} \tag{12.106}$$

These equations are easily solvable. Note that the equations for $\langle\hat{\hat{A}}_{\rm gi}\rangle$ and $\langle\hat{\hat{A}}_{\rm ie}\rangle$ are coupled only when $\gamma_x \neq 0$.

The problem of resonance fluorescence from a three level atom in external field can also be dealt with by means of the methods developed in preceding chapters. That problem can be handeled analytically in the secular approximation. We refer the reader to [118] for details and further references.

13. Dissipative Field Dynamics

In this chapter we solve some standard master equations describing evolution of a single and a two-mode field interacting linearly or non linearly with a reservoir. The non linear absorption considered here is a two-photon process.

13.1 Down-Conversion in a Damped Cavity

Consider a nonlinear medium placed inside a cavity. Let it be pumped by an external field of frequency $2\omega_0$ such that it causes generation of two photons of frequency ω_0 each by the process of down-conversion. The hamiltonian (7.63) for the process of non degenerate two-photon down-conversion assumes the form

$$\hat{H}_{\rm tp} = \frac{\hbar}{2}\left[G^* \exp(2{\rm i}\omega_0 t)\hat{a}^2 + G\exp(-2{\rm i}\omega_0 t)\hat{a}^{\dagger 2}\right] \tag{13.1}$$

when the process is degenerate. We assume the cavity to be tuned to ω_0. Let the cavity mode in question be coupled linearly also to an external drive. The hamiltonian governing this process is

$$\hat{H}_{\rm ext} = \hbar\left[g^* \exp({\rm i}\omega_0 t)\hat{a} + g\exp(-{\rm i}\omega_0 t)\hat{a}^\dagger\right]. \tag{13.2}$$

Let the cavity mode interact also with a reservoir which may possibly be squeezed. We will see shortly how such a squeezed reservoir is generated. The interaction with squeezed reservoir is described by (8.82). We assume that the frequency of the pump driving the reservoir is same as the frequency ω_0 of the cavity mode. In the interaction picture defined by

$$\hat{\rho}_{\rm I}(t) = \exp({\rm i}\omega_0\hat{a}^\dagger\hat{a}t)\hat{\rho}(t)\exp(-{\rm i}\omega_0\hat{a}^\dagger\hat{a}t), \tag{13.3}$$

the density operator then obeys the master equation

$$\begin{aligned}\frac{\rm d}{{\rm d}t}\hat{\rho}_{\rm I} = &-{\rm i}\left[g^*\hat{a} + g\hat{a}^\dagger,\ \hat{\rho}_{\rm I}\right] - \frac{\rm i}{2}\left[G\hat{a}^{\dagger 2} + G^*\hat{a}^2,\ \hat{\rho}_{\rm I}\right]\\ &+\kappa(\bar{N}+1)\left[2\hat{a}\hat{\rho}_{\rm I}\hat{a}^\dagger - \hat{\rho}_{\rm I}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}_{\rm I}\right]\\ &+\kappa\bar{N}\left[2\hat{a}^\dagger\hat{\rho}_{\rm I}\hat{a} - \hat{\rho}_{\rm I}\hat{a}\hat{a}^\dagger - \hat{a}\hat{a}^\dagger\hat{\rho}_{\rm I}\right]\\ &+\kappa\left[\bar{M}\left(2\hat{a}\hat{\rho}_{\rm I}\hat{a} - \hat{\rho}_{\rm I}\hat{a}^2 - \hat{a}^2\hat{\rho}_{\rm I}\right) + {\rm h.c.}\right].\end{aligned} \tag{13.4}$$

We assume further that

$$\kappa \geq |G|. \tag{13.5}$$

See [119] for details of the theory of the process of down–conversion in a thermal bath ($\bar{M} = 0$). We solve (13.4) by converting it in to an equation for $\rho(\alpha, \alpha^*) \equiv \langle\alpha|\hat{\rho}|\alpha\rangle$ by taking its matrix element in the coherent state $|\alpha\rangle$ and by using the relations (4.22)-(4.23). It reads

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\rho_{\mathrm{I}}(\alpha,\alpha^*) \\
&= -\mathrm{i}\left[g^*\frac{\partial}{\partial\alpha^*} - g\frac{\partial}{\partial\alpha}\right]\rho_{\mathrm{I}}(\alpha,\alpha^*) - \mathrm{i}\left[G^*\frac{\partial}{\partial\alpha^*}\alpha - G\frac{\partial}{\partial\alpha}\alpha^*\right]\rho_{\mathrm{I}}(\alpha,\alpha^*) \\
&\quad + \frac{1}{2}\left[-(\mathrm{i}G^* + 2\kappa\bar{M})\frac{\partial^2}{\partial\alpha^{*2}} + (\mathrm{i}G - 2\kappa\bar{M}^*)\frac{\partial^2}{\partial\alpha^2}\right]\rho_{\mathrm{I}}(\alpha,\alpha^*) \\
&\quad + \kappa\left[2(\bar{N}+1)\frac{\partial^2}{\partial\alpha\partial\alpha^*} + \frac{\partial}{\partial\alpha^*}\alpha^* + \frac{\partial}{\partial\alpha}\alpha\right]\rho_{\mathrm{I}}(\alpha,\alpha^*) \\
&\equiv \hat{\hat{L}}_1(\alpha,\alpha^*)\rho_{\mathrm{I}}(\alpha,\alpha^*).
\end{aligned} \tag{13.6}$$

We first examine the equations for the averages of α, α^*.

13.1.1 Averages and Variances of the Cavity Field Operators

Using (5.32) we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\langle\alpha\rangle\\ \langle\alpha^*\rangle\end{pmatrix} = \begin{pmatrix}-\kappa & -\mathrm{i}G\\ \mathrm{i}G^* & -\kappa\end{pmatrix}\begin{pmatrix}\langle\alpha\rangle\\ \langle\alpha^*\rangle\end{pmatrix} - \mathrm{i}\begin{pmatrix}g\\ -g^*\end{pmatrix}. \tag{13.7}$$

We solve these equations by applying the results of the Appendix C (C.5) and find that ($G = |G|\exp(\mathrm{i}\phi)$)

$$\begin{aligned}
&\begin{pmatrix}\langle\alpha(t)\rangle\\ \langle\alpha^*(t)\rangle\end{pmatrix} \\
&= \exp(-\kappa t)\begin{pmatrix}\cosh(|G|t) & -\mathrm{i}\exp(\mathrm{i}\phi)\sinh(|G|t)\\ \mathrm{i}\exp(-\mathrm{i}\phi)\sinh(|G|t) & \cosh(|G|t)\end{pmatrix} \\
&\quad \times\left[\begin{pmatrix}\langle\alpha(0)\rangle\\ \langle\alpha^*(0)\rangle\end{pmatrix} + \mathrm{i}|S\rangle\right] - \mathrm{i}|S\rangle,
\end{aligned} \tag{13.8}$$

$$|S\rangle = \frac{1}{\kappa^2 - |G|^2}\begin{pmatrix}\kappa g + \mathrm{i}Gg^*\\ -\kappa g^* + \mathrm{i}G^*g\end{pmatrix}. \tag{13.9}$$

Note that the time evolution is governed by $\exp[-(\kappa \pm |G|)t]$. This, by virtue of (13.5), describes damped evolution. As a result, the system decays towards a steady state given by the last term in (13.8).

Next we examine the evolution of the variances $\Delta\alpha^2, \Delta\alpha^{*2}, \Delta|\alpha|^2$ in $\alpha^2, \alpha^{*2}, |\alpha|^2$. It is straightforward to show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta\alpha^2 = -2\kappa\Delta\alpha^2 - 2\mathrm{i}G\Delta|\alpha|^2 + \mathrm{i}G - 2\kappa\bar{M}^*,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta\alpha^{*2} = -2\kappa\Delta\alpha^{*2} + 2\mathrm{i}G^*\Delta|\alpha|^2 - \mathrm{i}G^* - 2\kappa\bar{M},$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta|\alpha|^2 = -2\kappa|\alpha|^2 + \mathrm{i}(G^*\Delta\alpha^2 - G\Delta\alpha^{*2}) + 2\kappa(\bar{N}+1). \tag{13.10}$$

These equations can be combined to see that the equation for $G^*\Delta\alpha^2 + G\Delta\alpha^{*2}$ obeys a first–order equation which is solved by

$$G^*\Delta\alpha^2(t) + G\Delta\alpha^{*2}(t) = \exp(-2\kappa t)\left\{G^*\Delta\alpha^2(0) + G\Delta\alpha^{*2}(0)\right\} - (1-\exp(-2\kappa t))\left[G^*\bar{M}^* + G\bar{M}\right]. \tag{13.11}$$

The equations for $G^*\Delta\alpha^2 - G\Delta\alpha^{*2}$ and $\Delta|\alpha|^2$ are coupled reading

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} G^*\Delta\alpha^2 - G\Delta\alpha^{*2} \\ \Delta|\alpha|^2 \end{pmatrix} = \begin{pmatrix} -2\kappa & -4\mathrm{i}|G|^2 \\ \mathrm{i} & -2\kappa \end{pmatrix}\begin{pmatrix} G^*\Delta\alpha^2 - G\Delta\alpha^{*2} \\ \Delta|\alpha|^2 \end{pmatrix} + \begin{pmatrix} 2\mathrm{i}|G|^2 - 2\kappa(G\bar{M} - G^*\bar{M}^*) \\ 2\kappa(\bar{N}+1) \end{pmatrix}. \tag{13.12}$$

We solve this set by recalling (C.5) and find that

$$\begin{pmatrix} G^*\Delta\alpha^2(t) - G\Delta\alpha^{*2}(t) \\ \Delta|\alpha|^2(t) \end{pmatrix} = \exp(-2\kappa t)\begin{pmatrix} \cosh(2|G|t) & -2\mathrm{i}|G|\sinh(2|G|t) \\ \mathrm{i}\sinh(2|G|t)/2|G| & \cosh(2|G|t) \end{pmatrix} \times \times \left[\begin{pmatrix} G^*\Delta\alpha^2(0) - G\Delta\alpha^{*2}(0) \\ \Delta|\alpha|^2(0) \end{pmatrix} + |B\rangle\right] - |B\rangle, \tag{13.13}$$

$$|B\rangle = \frac{1}{2(\kappa^2 - |G|^2)}\begin{pmatrix} 2\mathrm{i}\kappa|G|^2(2\bar{N}+1) + 2\kappa^2(G\bar{M} - G^*\bar{M}^*) \\ |G|^2 - 2\kappa^2(\bar{N}+1) + \mathrm{i}\kappa(G\bar{M} - G^*\bar{M}^*) \end{pmatrix}. \tag{13.14}$$

We restrict our attention now to a thermal reservoir. This corresponds to $\bar{M} \to 0$ and $\bar{N} \to \bar{n}$ where $\bar{n}$ is average number of thermal photons in the reservoir. From (13.11) and (13.13) it follows that, in the steady state,

$$\begin{pmatrix} \Delta\alpha^2(\infty) \\ \Delta|\alpha|^2(\infty) \end{pmatrix} = \frac{1}{2(\kappa^2 - |G|^2)}\begin{pmatrix} -\mathrm{i}G\kappa(2\bar{n}+1) \\ 2\kappa^2(\bar{n}+1) - |G|^2 \end{pmatrix}. \tag{13.15}$$

Note from (13.8) that if $g = 0$, i.e. if the cavity mode is undriven then $\langle\alpha(\infty)\rangle = \langle\alpha^*(\infty)\rangle = 0$. Assume that to be the case and compute the steady-state value of the squeezing parameter

$$S(\psi) = \frac{1}{4}\langle(\exp(-\mathrm{i}\psi)\hat{a} + \exp(\mathrm{i}\psi)\hat{a}^\dagger)^2\rangle. \tag{13.16}$$

We recall that $S = 1/4$ in a coherent state and that $S < 1/4$ implies squeezing. While evaluating $S(\psi)$, bear it in mind that since $\rho(\alpha, \alpha^*)$ is a Q-function,

the average $\langle \alpha^{*m}\alpha^n\rangle$ corresponds to $\langle \hat{a}^n\hat{a}^{\dagger m}\rangle$. Thus $\langle |\alpha|^2\rangle$ corresponds to $\langle \hat{a}\hat{a}^\dagger\rangle$. Verify in particular that if $2\psi+\phi=\pi/2$ then

$$S=\frac{\kappa+2\kappa\bar{n}}{4(\kappa+|G|)}. \tag{13.17}$$

This shows that the field is definitely squeezed if $\bar{n}=0$, $G\neq 0$ and that the presence of thermal photons spoils squeezing. Since $|G|$ is restricted by (13.5), it follows that the maximum value of squeezing for $\bar{n}=0$ is $1/8$. This amounts to 50% squeezing over the value $1/4$ of S in a coherent state.

We leave it to the reader to use the results derived above to obtain two-time correlation functions of $\hat{a}$ and $\hat{a}^\dagger$. It will be found that the correlations $\langle \hat{a}(t)\hat{a}(t+\tau)\rangle$ and $\langle \hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau)\rangle$ are proportional to $|G|$. Furthermore, use (A.8) to show that the two-time correlations reduce to delta-functions in time as $\kappa\to\infty$.

We thus find that the field generated in the process of down-conversion in an ordinary bath is squeezed provided the number of thermal photons is low enough. The output field is also squeezed. The exact relationship between the cavity field and the output field requires further deliberations. We refer the reader to [93], [120]–[123] for a detailed account of the relationship between the cavity and the outcoming fields. The correlations in the output field are the same as in (8.65) with $G\to\xi$.

We will see that the knowledge of the averages and the variances derived above is sufficient to construct complete solution of the master equation.

13.1.2 Density Matrix

In this section we solve (13.6) for $\rho(\alpha,\alpha^*,t)$.

The formal solution of (13.6) is

$$\rho(\alpha,\alpha^*,t)=\exp\left(\hat{\hat{L}}_1(\alpha,\alpha^*)t\right)\rho(\alpha,\alpha^*,0). \tag{13.18}$$

Express $\rho(\alpha,\alpha^*,0)$ as

$$\rho(\alpha,\alpha^*,0)=\int \mathrm{d}^2\alpha_0\rho(\alpha_0,\alpha_0^*,0)\delta^{(2)}(\alpha-\alpha_0). \tag{13.19}$$

On combining this with (13.18) we obtain

$$\rho(\alpha,\alpha^*,t)=\int \mathrm{d}^2\alpha_0\ K(\alpha,\alpha^*,t;\alpha_0,\alpha_0^*,0)\rho(\alpha_0,\alpha_0^*,0), \tag{13.20}$$

where the kernel K is defined by

$$K(\alpha,\alpha^*,t;\alpha_0,\alpha_0^*,0)=\exp\left(\hat{\hat{L}}_1(\alpha,\alpha^*)t\right)\delta^{(2)}(\alpha-\alpha_0). \tag{13.21}$$

On using the representation (A.2) of the delta-function, (13.21) reads

$$\begin{aligned}&K(\alpha,\alpha^*,t;\alpha_0,\alpha_0^*,0)\\&=\frac{1}{\pi^2}\int \mathrm{d}^2\xi\,\exp[-\mathrm{i}(\xi\alpha_0+\xi^*\alpha_0^*)]\exp\left(\hat{\hat{L}}_1 t\right)\exp[\mathrm{i}(\xi\alpha+\xi^*\alpha^*)].\end{aligned} \tag{13.22}$$

Now, note that $\hat{\tilde{L}}_1(\alpha,\alpha^*)$ is a linear combination of the operators $\partial/\partial\alpha$, $\alpha\partial/\partial\alpha$, $\alpha^*\partial/\partial\alpha$, $\partial^2/\partial\alpha^2$, $\partial^2/\partial\alpha\partial\alpha^*$ and their complex conjugates and that this set of operators is closed under the operation of commutation. Hence, the exponential in (13.22) can be expressed as a product of the exponentials of the said operators. The scalar coefficients in each of the exponents may be determined by the method outlined in the Chap. 2. There is, however, no need to follow this tortuous path! Instead, we observe that the exponentials of the said operators acting on the exponential function on the right in (13.22) reduce it to the form

$$K(\alpha,\alpha^*,t;\alpha_0,\alpha_0^*,0)=\frac{1}{\pi^2}\int \mathrm{d}^2\xi \exp\left(B\xi^{*2}+B^*\xi^2-d|\xi|^2\right) \times \exp\left[\mathrm{i}\xi\left(f_1\alpha+f_2\alpha^*-\alpha_0+f_0\right)-\text{c.c.}\right]. \quad (13.23)$$

This shows that, if the coefficients of $\xi,\xi^*,|\xi|^2$ satisfy appropriate conditions for the integral in (13.23) to exist then the kernel is a Gaussian in α,α^*. Now, recall that a Gaussian is expressible in terms of averages and correlations of the variables as in (5.17). Since the variables in that equation are real, we apply it for the real and imaginary parts, $x_1=(\alpha+\alpha^*)/2$ and $x_2=(\alpha-\alpha^*)/2\mathrm{i}$ of α and express the Gaussian kernel as

$$K(\alpha,\alpha^*,t;\alpha_0,\alpha_0^*,0)=\frac{1}{\pi\sqrt{\chi}}\exp\left[-\frac{1}{8\chi}\boldsymbol{V}^T\hat{M}\boldsymbol{V}\right], \quad (13.24)$$

$$\boldsymbol{V}=\begin{pmatrix}\alpha-\langle\alpha\rangle+\text{c.c.}\\ -\mathrm{i}\left(\alpha-\langle\alpha\rangle-\text{c.c.}\right)\end{pmatrix},$$

$$\hat{M}=\begin{pmatrix}-\Delta\left(\alpha-\alpha^*\right)^2 & \mathrm{i}\Delta\left(\alpha^2-\alpha^{*2}\right)\\ \mathrm{i}\Delta\left(\alpha^2-\alpha^{*2}\right) & \Delta\left(\alpha+\alpha^*\right)^2\end{pmatrix}, \quad (13.25)$$

$\boldsymbol{V}^T$ being the transpose of $\boldsymbol{V}$ and

$$\chi=(\Delta|\alpha|^2)^2-\Delta\alpha^2\Delta\alpha^{*2}. \quad (13.26)$$

The averages and the variances in the expressions above are given by (13.8), (13.11) and (13.13) with the understanding that the variables α,α^* have fixed values α_0,α_0^* at $t=0$. Hence, $\langle\alpha(0)\rangle=\alpha_0$ in (13.8) whereas $\Delta\alpha^2(0)=\Delta\alpha^{*2}(0)=\Delta|\alpha|^2(0)=0$ in (13.11) and (13.13).

We have thus at hand the expression for the Q-function corresponding to the density operator satisfying (13.4). The kernel for any other quasidistribution would also have the form (13.24) except that the equations for the averages and the variances are to be derived from the master equation corresponding to the desired quasiprobability. Alternatively, use the relationship (4.12) between the quantities in question for different quasiprobabilities. The averages of α,α^* are same for all the quasiprobabilities but the average of $|\alpha|^2$ is different. For example, recall that $\langle|\alpha|^2\rangle$ represents $\langle\hat{a}\hat{a}^\dagger\rangle$, $\langle\hat{a}^\dagger\hat{a}\rangle$ and $\langle\hat{a}\hat{a}^\dagger+\hat{a}\hat{a}^\dagger\rangle/2$, respectively for the Q, P and the Wigner functions. Hence, the expression for $\rho(\alpha,\alpha^*)$ stands for any ordering by assigning the value to $\Delta|\alpha|^2$ appropriate to that ordering.

Density Matrix in Number State Representation. In many applications, we need matrix elements of the density operator in the number states. These can be derived by noting that

$$\langle m|\hat{\rho}|n\rangle = \frac{1}{\sqrt{m!n!}} \frac{\partial^{m+n}}{\partial\alpha^{*m}\partial\alpha^{n}} \exp(|\alpha|^2)\hat{\rho}(\alpha,\alpha^*)\Big|_{\alpha=\alpha^*=0}. \tag{13.27}$$

As an illustration, we consider a thermal bath characterized by $\bar{M} = 0$, $\bar{N} = \bar{n}$. We also let $G = 0$ and find that

$$\begin{aligned} &\langle\alpha\rangle = d\alpha_0, \quad \Delta\alpha^2 = 0, \\ &\chi = \Delta|\alpha|^2 = (\bar{n}+1)(1-d^2), \qquad d = \exp(-\kappa t). \end{aligned} \tag{13.28}$$

Evaluate the kernel by substituting these expressions in (13.24)-(13.26). Combine the result with (13.20) to show that

$$\begin{aligned} &\rho(\alpha,\alpha^*,t) \\ &= \frac{1}{\pi\chi}\int \mathrm{d}^2\alpha_0 \exp\Big[-\Big|\alpha - \exp(-\kappa t)\alpha_0\Big|^2/\chi\Big]\rho(\alpha_0,\alpha_0^*,0). \end{aligned} \tag{13.29}$$

Now we evaluate $\langle m|\hat{\rho}|n\rangle$ by substituting this in (13.27). The operation of differentiation leads to

$$\begin{aligned} &\frac{\partial^m}{\partial\alpha^{*m}} \exp\left[\frac{\alpha_0 d(\alpha^* - \alpha_0^* d)}{\chi}\right] \frac{\partial^n}{\partial\alpha^n} \exp\left[-\frac{\alpha}{\chi}\{\alpha^*(1-\chi) - \alpha_0^* d\}\right]\Big|_{\alpha=\alpha^*=0} \\ &= (-\chi)^{-n} \exp\left(-\frac{|\alpha_0|^2 d^2}{\chi}\right) \frac{\partial^m}{\partial\alpha^{*m}} \exp(\alpha_0\alpha^* d/\chi)\{\alpha^*(1-\chi) - \alpha_0^* d\}^n\Big|_{\alpha^*=0} \\ &= (\chi)^{-n} \exp(-|\alpha_0|^2 d^2/\chi) \sum_k (-)^k \Big[\frac{m!n!}{(m-k)!(n-k)!k!}(1-\chi)^k \\ &\quad \times(\alpha_0^* d)^{n-k}(\alpha_0 d/\chi)^{m-k}\Big]. \end{aligned} \tag{13.30}$$

Now, rewrite $\hat{\rho}(\alpha_0,\alpha_0^*,0)$ as

$$\hat{\rho}(\alpha_0,\alpha_0^*,0) = \exp(-|\alpha_0|^2) \sum_{p,q} \frac{1}{\sqrt{p!q!}} \alpha_0^{*p}\alpha_0^q \langle p|\hat{\rho}(0)|q\rangle. \tag{13.31}$$

Substitute this, along with (13.30) in (13.27). Carry the integration over α_0, α_0^* assuming $m \geq n$. Change the summation over k in (13.30) to $n-k$. Rewrite the resulting sum over k in terms of the hypergeometric function to obtain

$$\begin{aligned} \langle m|\hat{\rho}(t)|n\rangle = &\sqrt{\frac{m!}{n!}} \frac{1}{\chi+d^2}(1-\chi^{-1})^n(\chi+d^2)^{n-m}d^{m-n} \times \\ &\times \sum_l \chi^l \Bigg[\sqrt{\frac{(m-n+l)!}{l!}} \frac{\langle m-n+l|\hat{\rho}(0)|l\rangle}{(\chi+d^2)^l} \\ &\times F(m-n+l+1,-n;m-n+1;x)\Bigg], \end{aligned} \tag{13.32}$$

$$x = d^2/(\chi + d^2)(1 - \chi). \tag{13.33}$$

If $\bar{n} = 0$ then we note that $x = 1$. The value of the hypergeometric function $F(l + 1, -m, n - m + 1, 1)$ is given by (A.38). The expression (13.32) then assumes the form

$$\begin{aligned}&\langle m|\hat{\rho}|m\rangle \\ &= \exp(-2\kappa mt) \sum_p \frac{p!}{m!(p-m)!}(1 - \exp(-2\kappa t))^{p-m}\langle p|\hat{\rho}|p\rangle.\end{aligned} \tag{13.34}$$

We have employed this expression in Chap. 14 in the context of the theory of a micromaser.

13.2 Field Interacting with a Two-Photon Reservoir

In this section we discuss the method of solving the master equation of a single or a two-mode field interacting with a two-photon reservoir. We consider first the evolution of the field only under the process of two-photon absorption. We follow it up by including a mechanism that generates the field in a two-photon process.

13.2.1 Two-Photon Absorption

The interaction of a two-mode field with two-photon reservoir is described by the master equations (8.116) whereas that of a single-mode field is described by replacing $\hat{b}$ by $\hat{a}$ in that equation. By virtue of the relations (2.71) and (2.76) the master equation for both the cases can be expressed in terms of the $SU(1,1)$ operators $\hat{K}_\pm, \hat{K}_z$. In the interaction picture generated by the free-field hamiltonian, those equations are of the form

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{\rho} = &-\mathrm{i}\eta\left[\hat{K}_+\hat{K}_-,\ \hat{\rho}\right] + \kappa(\bar{n}+1)\left[2\hat{K}_-\hat{\rho}\hat{K}_+ - \hat{\rho}\hat{K}_+\hat{K}_- - \hat{K}_+\hat{K}_-\hat{\rho}\right] \\ &+\kappa\bar{n}\left[2\hat{K}_+\hat{\rho}\hat{K}_- - \hat{\rho}\hat{K}_-\hat{K}_+ - \hat{K}_-\hat{K}_+\hat{\rho}\right].\end{aligned} \tag{13.35}$$

It is convenient to employ the eigenstates $\{|\chi(m,K)\rangle\}$ of $\hat{K}_z$ for a fixed eigenvalue label K of $\hat{Q}$ as a basis. However, we may require the expectation value of such combinations of field operators that do not commute with $\hat{Q}$. We, therefore, need to know $\langle\chi(m,K_2)|\hat{\rho}|\chi(n,K_1)\rangle$ even for $K_1 \neq K_2$.

We also observe that if we take the matrix element of (13.35) in the states $|\chi(m,K_1)\rangle$ and $|\chi(n,K_2)\rangle$ then the value of $m - n$ in each term is same.

Bearing these observations in mind, we solve (13.35) by taking its matrix elements between the states $|\chi(m,K_1)\rangle$ and $|\chi(m+p,K_2)\rangle$ where K_1 and K_2 may be different. The corresponding eigenvalue equation, in terms of the function

$$F_m^{(p)} = \sqrt{\frac{\Gamma(K_1+m)\Gamma(K_2+m+p)}{\Gamma(m+1)\Gamma(m+p+1)}}\langle\chi(m,K_1)|\hat{\rho}|\chi(m+p,K_2)\rangle \quad (13.36)$$

reads ($\kappa = 1$)

$$\left[(1-\mathrm{i}\eta)\mu_{m+p}^2(K_2)+(1+\mathrm{i}\eta)\mu_m^2(K_1)-\bar{n}\left\{\mu_{m+p}^2+\mu_m^2\right\}-\lambda^{(p)}\right]F_m^{(p)}$$
$$2(\bar{n}+1)(K_1+m+1)(K_2+m+p+1)F_{m+1}^{(p)}$$
$$2\bar{n}(K_1+m+1)(K_2+m+p+1)F_{m-1}^{(p)}=0, \quad (13.37)$$

$$\mu_m(K) = \sqrt{m(2K+m-1)}. \quad (13.38)$$

On comparing this with the standard form (10.41) we find that this equation does not fall in to any analytically solvable class unless $\bar{n} = 0$. If $\bar{n} = 0$ then (13.37) reduces to the two-term recursion relation

$$\alpha_{m+1,p}\rho_{m+1,p} - \beta_{m,p}\rho_{m,p} = \lambda^{(p)}\rho_{m,p} \quad (13.39)$$

for $\rho_{m,p} \equiv \langle\chi(m,K_1)|\hat{\rho}|\chi(m+p,K_2)\rangle$ with

$$\alpha_{m,p} = 2\mu_{m+p}(K_2)\mu_m(K_1),$$
$$\beta_{m,p} = (1-\mathrm{i}\eta)\mu_{m+p}^2(K_2)+(1+\mathrm{i}\eta)\mu_m^2(K_1). \quad (13.40)$$

By following Sec.10.3, we note that the eigenvalues are given by

$$\lambda_n^{(p)} = -\beta_n, p. \quad (13.41)$$

Verify also that the eigenvalues are distinct and that their corresponding eigenvectors are

$$\rho_{m,p}^{(n)} = 0, \quad m \geq n+1,$$
$$\rho_{m,p}^{(n)} = \frac{(C_p+2n)\,\Gamma(C_p+m+n)}{(n-m)!\sqrt{m!(m+p)!\Gamma(2K_1+m)\,\Gamma(2K_2+m+p)}}, \quad (13.42)$$

for $m \leq n$, with $(C_p + 2n)$ as the normalization factor (see (13.46) below),

$$C_p = K_1 + K_2 + p - \mathrm{i}\eta(K_1 - K_2 - p). \quad (13.43)$$

Note that the adjoint of (13.39) is

$$\alpha_{m,p}\rho_{m-1,p} - \beta_{m,p}^*\rho_{m,p} = \lambda^{*(p)}\rho_{m,p} \quad (13.44)$$

Following Sect. 10.4, the eigenvectors $\tilde{\rho}_{m,p}^{(n)}$ are given by

$$\tilde{\rho}_{m,p}^{(n)} = 0, \quad m \leq n-1,$$
$$\tilde{\rho}_{m+n,p}^{(n)} = (-)^m\frac{\sqrt{(m+n)!(m+n+p)!}}{m!\Gamma(C_p+2n+m+1)}$$
$$\times\sqrt{\Gamma(2K_1+m+n)\Gamma(2K_2+m+n+p)}. \quad (13.45)$$

We ascertain the orthogonality of the sets (13.42) and (13.45) by evaluating

$$\sum_{m=0} \rho_{m,p}^{(n)} \tilde{\rho}_{m,p}^{(k)} = \sum_{m=k}^{n} \rho_{m,p}^{(n)} \tilde{\rho}_{m,p}^{(k)} = \sum_{m=0}^{n} \rho_{k+m,p}^{(n-k)} \tilde{\rho}_{k+m,p}^{(k)}$$

$$= (C_p + 2n) \sum_{m=0}^{n-k} (-)^m \frac{\Gamma(C_p + m + n + k)}{m!(n-m-k)!\Gamma(C_p + 2k + m + 1)}$$

$$= \frac{(C_p + 2n)\,\Gamma(C_p + n + k)}{\Gamma(C_p + 2k + 1)\,(n-k)!}$$

$$\times F(C_p + n + k, -n + k; C_p + 2k + 1; 1)$$

$$= \frac{(C_p + 2n)\,\Gamma(C_p + n + k)}{(n-k)!(k-n)!\Gamma(C_p + n + k + 1)}$$

$$= \delta_{kn}. \tag{13.46}$$

The density matrix at any time t is then given by

$$\langle \chi(m)|\hat{\rho}(t)|\chi(m+p)\rangle = \sum_{l,n=0} \left[\exp\left(-\lambda_n^{(p)} t\right) \rho_{m,p}^{n} \tilde{\rho}_{l,p}^{n} \right.$$

$$\left. \times \langle \chi(m, K_1)|\hat{\rho}(0)|\chi(m+p, K_2)\rangle. \right. \tag{13.47}$$

As explained above, this enables us to obtain the expectation value of any field operator whether it is an $SU(1,1)$ operator or not. For a different method for evaluating $\hat{\rho}(t)$ for $\eta = 0$, and for some numerical results, see [95, 124].

13.2.2 Two-Photon Generation and Absorption

Consider now the situation wherein the field, in addition to being absorbed in a two-photon reservoir, is generated by another two-photon mechanism. As discussed in [55], this situation depicts closely the experiments of [125]. Assuming that the frequency of the field driving the nonlinear medium to produce two photons is equal to the sum of the frequencies of the photons produced, the hamiltonian describing two-photon generation in the interaction picture is given by

$$\hat{H}_1 = \frac{\hbar}{2}[G\hat{a}^2 + G^*\hat{a}^{\dagger 2}], \tag{13.48}$$

for the single-mode field and by

$$\hat{H}_2 = \hbar[G\hat{a}\hat{b} + G^*\hat{b}^\dagger\hat{a}^\dagger], \tag{13.49}$$

for the two-mode field. These hamiltonians consist only of $SU(1,1)$ operators. The master equation for the field, with (13.35) as the damping part (with $\bar{n} = 0$), may be written as ($\kappa = 1$)

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{\rho} = 2\hat{\tilde{K}}_-\hat{\rho}\hat{\tilde{K}}_+ - (1 + \mathrm{i}\eta)\hat{\rho}\hat{\tilde{K}}_+\hat{\tilde{K}}_- - (1 - \mathrm{i}\eta)\hat{\tilde{K}}_+\hat{\tilde{K}}_-\hat{\rho}, \tag{13.50}$$

$$\hat{\tilde{K}}_- = \hat{K}_- + \frac{\mathrm{i}G}{1 - \mathrm{i}\eta} = \hat{\tilde{K}}_+^\dagger. \tag{13.51}$$

If there exists a state $|\psi_0\rangle$ such that $\hat{\hat{K}}_-|\psi_0\rangle = 0$, i.e. if

$$\left[\hat{K}_- + \frac{\mathrm{i}G}{1-\mathrm{i}\eta}\right]|\psi_0\rangle = 0 \tag{13.52}$$

then

$$\hat{\rho}_{\mathrm{ss}} = |\psi_0\rangle\langle\psi_0| \tag{13.53}$$

is the steady state solution of (13.50). Equation (13.52) is the same as (3.120) whose solution is given by (3.121). For single-mode realization, (13.52) defines the pair coherent states [55]. For a discussion of numerical results, see [55].

13.3 Reservoir in the Lambda Configuration

The master equation for a single-mode field interacting with a reservoir in the Raman configuration is given by replacing $\hat{b}$ by $\hat{a}$ in (8.118). It is a function of only the number operators $\hat{a}^\dagger\hat{a}$. It is trivially solvable in the number states representation.

We consider the two-mode case dissipating according to (8.118). Let there be a nonlinear medium pumped by an external field such that it gnerates and absorbs the modes in question in the Lambda configuration. The modes in the interaction picture then evolve according to the master equation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho} = &-\mathrm{i}\epsilon\left[\hat{a}^\dagger\hat{b} + \hat{b}^\dagger a,\ \hat{\rho}\right] + \kappa(\bar{n}+1)\left[2\hat{a}^\dagger\hat{b}\hat{\rho}\hat{b}^\dagger\hat{a} - \hat{\rho}\hat{b}^\dagger\hat{a}\hat{a}^\dagger\hat{b} - \hat{b}^\dagger\hat{a}\hat{a}^\dagger\hat{b}\hat{\rho}\right] \\ &+\kappa\bar{n}\left[2\hat{b}^\dagger\hat{a}\hat{\rho}\hat{a}^\dagger\hat{b} - \hat{\rho}\hat{a}^\dagger\hat{b}\hat{b}^\dagger\hat{a} - \hat{a}^\dagger\hat{b}\hat{b}^\dagger\hat{a}\hat{\rho}\right].\end{aligned} \tag{13.54}$$

We may use the eigenstates $\{|m, N-m\rangle\}$, corresponding respectively to the eigenvalues m and $N-m$ of $\hat{a}^\dagger\hat{a}$ and $\hat{b}^\dagger\hat{b}$ as the basis states. By following the arguments similar to the ones in the last section, we note that we need to evaluate the matrix elements of $\hat{\rho}$ between states of different values of N. We also observe that if we take matrix elements of (13.54) in the states $|m, N_1 - m\rangle$ and $|n, N_2 - n\rangle$ then each term in (13.54) preserves the value of $m-n$.

We, therefore, take matrix elements of (13.54) in the states $|m, N_1 - m\rangle$ and $|m+p, N_2 - m - p\rangle$. By following the method of the last section, we find that the analytical solution of (13.54) may be derived by the method of Chap. 10 if $\bar{n} = 0$. Assuming that to be the case, the eigenvalue equation corresponding to (13.54) reads ($\epsilon = 0, \kappa = 1$)

$$\alpha_{mp}^{(n)}\rho_{m-1p} - \beta_{m,p}\rho_{mp}^{(n)} = \lambda_n^{(p)}\rho_{mp}^{(n)}, \tag{13.55}$$

$$\alpha_{mp} = 2\sqrt{m(m+p)(N_1-m+1)(N_2-m-p+1)},$$
$$\beta_{mp} = 2m\left(\frac{N_1+N_2}{2}-m-p-1\right)+p\,(N_2-p-1)+N_1+N_2$$
$$+\mathrm{i}\eta\left[m\,(N_1-N_2+2p)-p\,(N_2-p)\right]. \tag{13.56}$$

By following Sect. 10.3 we note that the eigenvalues are given by

$$\lambda_n^{(p)} = -\beta_{np}. \tag{13.57}$$

The corresponding eigenvectors being

$$\rho_{mp}^{(n)} = 0, \quad m \le n-1,$$
$$\rho_{mp}^{(n)} = A_n \frac{(-)^m}{(m-n)!\Gamma\left(m+n+p-S-\mathrm{i}D_p\right)}$$
$$\times\sqrt{\frac{m!(m+p)!}{(N_1-m)!(N_2-m-p)!}}, \quad m \ge n, \tag{13.58}$$

where A_n is the normalization constant and

$$S = \frac{N_1+N_2}{2}-1, \qquad D_p = \eta\left(\frac{N_1-N_2}{2}+p\right). \tag{13.59}$$

It may also be confirmed that the eigenvectors of the equation

$$\alpha_{m+1p}^{(n)}\tilde{\rho}_{m+1p} - \beta_{mp}^{*}\tilde{\rho}_{mp}^{(n)} = \lambda_n^{*(p)}\tilde{\rho}_{mp}^{(n)}, \tag{13.60}$$

adjoint to (13.55) are

$$\tilde{\rho}_{mp}^{(n)} = 0, \quad m \ge n+1,$$
$$\tilde{\rho}_{mp}^{(n)} = \frac{(-)^m}{(n-m)!\Gamma\left(S-p-m-n+1+\mathrm{i}D_p\right)}$$
$$\times\sqrt{\frac{(N_1-m)!(N_2-m-p)!}{m!(m+p)!}}, \quad m \le n. \tag{13.61}$$

The scalar product between the eigenvectors (13.58) and (13.61) is

$$\sum_{m=0}\rho_{m,p}^{(k)}\tilde{\rho}_{m,p}^{(n)} = \sum_{m=k}^{n}\tilde{\rho}_{m_p}^{(n)}\rho_{m_p}^{(k)}$$
$$= \frac{A_n}{\Gamma\left(2k-S+p-1-\mathrm{i}D_p\right)(n-k)!\Gamma\left(S-p-1+\mathrm{i}D_p-n-k+2\right)}$$
$$\times F\left(-n+k, n+k-S+p-1-\mathrm{i}D_p; 2k-S-p-1-\mathrm{i}D_p; 1\right)$$
$$= \frac{A_n}{\Gamma\left(S-p+1+\mathrm{i}D_p-2n\right)\Gamma\left(2n-S+p-\mathrm{i}D_p\right)}\delta nk. \tag{13.62}$$

This determines the normalization constant. As discussed in Chap. 10, these eigenvectors are sufficient to determine the solution of (13.54) for a given p if the corresponding eigenvalues are nondegenerate and that we need to find

the generalized eigenvectors in case of degeneracy. We note that, for $N_1 = N_2 = N$, $\lambda_n^{(0)} = \lambda_{N-n-1}^{(0)}$ if $p = 0$, i.e. the eigenvalues are two-fold degenerate. The solution of the equation in this case requires a knowledge of the generalized eigenvectors corresponding to two-fold degenerate eigenvalues. However, we do not undertake the tedious but straightforward task of construction of generalized eigenvectors.

14. Dissipative Cavity QED

In this chapter we illustrate the general methods developed in previous chapters by applying them to a dissipative atomic system in a dissipative cavity. Except for atoms modeled as harmonic oscillators, this system can not generally be treated analytically exactly. In this chapter we consider a system of two-level atoms. We pay attention to theoretically and experimentally interesting situation of strong atom-field coupling. We discuss the ways of probing the characteristic features of such a strongly coupled atoms-cavity system by means of an external drive. Finally, we outline the method of treating the situation wherein atoms are pumped in to a cavity one at a time. This constitutes what is known as a micromaser.

14.1 Two-Level Atoms in a Single-Mode Cavity

We consider a system of N identical two-level atoms of transition frequency ω_0 coupled to a cavity mode of frequency ω. The corresponding hamiltonian is

$$\hat{H} = \hbar\left[\omega\hat{a}^\dagger\hat{a} + \omega_0\hat{S}_z\right] + \hat{H}_{\mathrm{a-f}}, \tag{14.1}$$

$$\hat{H}_{\mathrm{a-f}} = g\hbar\left[\hat{a}^\dagger\hat{S}_- + \hat{S}_+\hat{a}\right]. \tag{14.2}$$

We assume that the cavity mode in question and the atoms are coupled to a thermal bath having an average of $\bar{n}$ photons. The master equation for the density operator $\hat{\rho}$ of the combined system of the atoms and the cavity mode in the interaction representation generated by $\omega(\hat{a}^\dagger\hat{a} + \hat{S}_z)$ then reads (with $\delta = \omega_0 - \omega$)

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho} = -\frac{\mathrm{i}}{\hbar}[\hat{H}_0, \hat{\rho}] + \hat{\hat{\Lambda}}_{\mathrm{f}}\hat{\rho} + \hat{\hat{\Lambda}}_{\mathrm{a}}\hat{\rho}, \equiv \hat{\hat{L}}\hat{\rho}, \tag{14.3}$$

$$\hat{H}_0 = \delta\hat{S}_z + \hat{H}_{\mathrm{a-f}}, \tag{14.4}$$

$$\begin{aligned}\hat{\hat{\Lambda}}_{\mathrm{f}} = {} & \kappa(\bar{n}+1)\left[2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\right] \\ & +\kappa\bar{n}\left[2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{a}^\dagger - \hat{a}\hat{a}^\dagger\hat{\rho}\right],\end{aligned} \tag{14.5}$$

$$\hat{\Lambda}_{\rm a} = \gamma(\bar{n}+1)\left[2\hat{S}_-\hat{\rho}\hat{S}_+ - \hat{\rho}\hat{S}_+\hat{S}_- - \hat{S}_+\hat{S}_-\hat{\rho}\right] + \gamma\bar{n}\left[2\hat{S}_+\hat{\rho}\hat{S}_- - \hat{\rho}\hat{S}_-\hat{S}_+ - \hat{S}_-\hat{S}_+\hat{\rho}\right]. \tag{14.6}$$

The equation (14.3) may be solved by converting it in to a c-number equation by taking its matrix elements between the the states $\{|m, M\rangle\}$ where $|m\rangle$ is a number state of the of the field and $|M\rangle$ ($M = -N/2, -N/2+1, \cdots, N/2$) is an eigenstate of $\hat{S}_z$. Its steady state solution can be found by applying the principle of detailed balance (see Sect. 12.2). Else, verify by direct substitution that if $\bar{n} = 0$ then the steady state is given by

$$\hat{\rho}_{\rm ss} = |0, -N/2\rangle\langle 0, -N/2|. \tag{14.7}$$

The steady state for $\bar{n} \neq 0$ reads

$$\hat{\rho}_{\rm ss} = \exp[-\beta(\hat{a}^\dagger\hat{a} + \hat{S}_z)], \qquad \exp(-\beta) = \frac{\bar{n}}{\bar{n}+1}. \tag{14.8}$$

The time-dependent solution of (14.3) even for a single two-level atom is a formidable task except when $\bar{n} = 0$ and the eigenvalue of $\hat{a}^\dagger\hat{a} + \hat{S}_z$ in the initial state is $\pm 1/2$ [127].

However, some situations of practical interest in quantum optics that can be tackled much more conveniently by approximation methods include (i) the limit of strong atom-field coupling compared with dissipation, and (ii) weak atomic coupling with an external drive. The former enables us to invoke the secular approximation whereas the latter can be handled perturbatively.

14.2 Strong Atom–Field Coupling

The situations in which the reversible atom-cavity coupling is very much stronger than irreversible dissipation are of immense interest. Recall from Sect. 8.2 that such situations can be treated conveniently by the method of the secular approximation. We outline this method (i) for a single two-level atom when the atom and the field are in an arbitrary initial state, and (ii) for N two-level atoms in particular initial state.

14.2.1 Single Two-Level Atom

Recall that the secular approximation consist in retaining only those terms in the dissipative operator which are time-independent in the interaction picture generated by the strong reversible part. The reversible part in the present case is the hamiltonian $\hat{H}_0$. We express the density operator of the system as a linear combination of the eigenstates of $\hat{H}_0$. For the sake of illustration we let $\delta = 0$. Corresponding eigenstates $|\psi_m^{(\pm)}\rangle$ and the eigenvalues $\mu_m^{(\pm)} = \pm g\sqrt{m+1}$ of $\hat{H}_0$ are given respectively by (11.32) and (11.33). The density operator may then be expressed as

$$\hat{\rho} = |0,-1/2\rangle\langle 0,-1/2| + \sum_{m=0}\sum_{\alpha=\pm}\{C_m^{(\alpha)}|0,-1/2\rangle\langle\psi_m^{(\pm)}| + \text{h.c.}\}$$
$$+ \sum_{m,n=0}\sum_{\alpha,\beta=\pm} C_{mn}^{(\alpha,\beta)}|\psi_m^{(\alpha)}\rangle\langle\psi_n^{(\beta)}|. \tag{14.9}$$

We, therefore, derive the equation for $|\psi_m^{(\alpha)}\rangle\langle\psi_n^{(\beta)}|$ by letting $\hat{\rho} \to |\psi_m^{(\alpha)}\rangle\langle\psi_n^{(\beta)}|$ in (14.3). The secular approximation consists in retaining only those terms in $[\hat{\Lambda}_a + \hat{\Lambda}_f]|\psi_m^{(\alpha)}\rangle\langle\psi_n^{(\beta)}|$ which have same time dependence as $|\psi_m^{(\alpha)}\rangle\langle\psi_n^{(\beta)}|$ under the action of the evolution generated by $\hat{H}_0$. This means that if $|\psi_p^{(\gamma)}\rangle\langle\psi_q^{(\delta)}|$ is a term in the expression for $[\hat{\Lambda}_a + \hat{\Lambda}_f]|\psi_m^{(\alpha)}\rangle\langle\psi_n^{(\beta)}|$ then it may be retained in the secular approximation if $\mu_p^{(\gamma)} - \mu_q^{(\delta)} = \mu_m^{(\alpha)} - \mu_n^{(\beta)}$. The action of the damping operator on a dressed state may be found by first writing the dressed state in terms of the bare states $|m, \pm 1/2\rangle$. Use the knowledge of the action of the operators on the bare states and rewrite the resulting bare states in terms of the dressed states. As an illustration, we determine the action of $\hat{a}$ on $|\psi_m^{(\pm)}\rangle$. Using (11.32), we find that, for $m = 1, 2, \ldots$,

$$\hat{a}|\psi_m^{(\pm)}\rangle = \frac{1}{\sqrt{2}}\hat{a}\left[|m,1/2\rangle \pm |m+1,-1/2\rangle\right]$$
$$= \frac{1}{\sqrt{2}}\left[\sqrt{m}|m-1,1/2\rangle \pm \sqrt{m+1}|m,-1/2\rangle\right]$$
$$= \left[\Gamma_m^{+}|\psi_{m-1}^{(\pm)}\rangle - \Gamma_m^{-}|\psi_{m-1}^{(\mp)}\rangle\right], \tag{14.10}$$

$$\Gamma_m^{(\pm)} = \frac{1}{2}\left[\sqrt{m+1} \pm \sqrt{m}\right], \tag{14.11}$$

$$\hat{a}|\psi_0^{(\pm)}\rangle = \pm\frac{1}{\sqrt{2}}|0,-1/2\rangle. \tag{14.12}$$

In the same manner show that

$$\hat{a}^\dagger\hat{a}|\psi_m^{(\pm)}\rangle = \frac{1}{2}\left[(2m+1)|\psi_m^{(\pm)}\rangle - |\psi_m^{(\mp)}\rangle\right]. \tag{14.13}$$

Bearing these considerations in mind, it straightforward to see that the master equation (14.3) for $\bar{n} = 0 = \gamma = 0$, $\omega = \omega_0$ leads to the following equations in the secular approximation

$$\frac{\mathrm{d}}{\mathrm{d}t}|\psi_m^{(\alpha)}\rangle\langle\psi_n^{(\beta)}| = -\left[\mathrm{i}(\mu_m^{(\alpha)} - \mu_n^{(\beta)}) + \kappa(m+n+1)\right]|\psi_m^{(\alpha)}\rangle\langle\psi_n^{(\beta)}|, \tag{14.14}$$

$m \neq n$ and any $\alpha, \beta = \pm$ or $\alpha \neq \beta$ and any m, n,

$$\frac{\mathrm{d}}{\mathrm{d}t}|0,-1/2\rangle\langle\psi_n^{(\beta)}| = \left[\mathrm{i}\mu^{(\beta)} - \kappa(n+1/2)\right]|0,-1/2\rangle\langle\psi_n^{(\beta)}|, \tag{14.15}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}|0,-1/2\rangle\langle 0,-1/2| = 0. \tag{14.16}$$

The equations for the diagonal components read, $m = 1, 2, \ldots$,

$$\frac{\mathrm{d}}{\mathrm{d}t}|\psi_m^{(+)}\rangle\langle\psi_m^{(+)}| = 2\kappa\Big[\Gamma_m^{(+)2}|\psi_{m-1}^{(+)}\rangle\langle\psi_{m-1}^{(+)}| + \Gamma_m^{(-)2}|\psi_{m-1}^{(-)}\rangle\langle\psi_{m-1}^{(-)}| - (m+1/2)|\psi_m^{(+)}\rangle\langle\psi_m^{(+)}|\Big], \tag{14.17}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}|\psi_m^{(-)}\rangle\langle\psi_m^{(-)}| = 2\kappa\Big[\Gamma_m^{(+)2}|\psi_{m-1}^{(-)}\rangle\langle\psi_{m-1}^{(-)}| + \Gamma_m^{(-)2}|\psi_{m-1}^{(+)}\rangle\langle\psi_{m-1}^{(+)}| - (m+1/2)|\psi_m^{(-)}\rangle\langle\psi_m^{(-)}|\Big]. \tag{14.18}$$

Whereas the equation (14.14) for off-diagonal elements is simple, the equations (14.17) and (14.18) constitute coupled set of recursion relations for the diagonal elements. Their sum and difference lead to the equations (with $\tau = 2\kappa t$)

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{F}_m^+(t) = -\left(m+\frac{1}{2}\right)\hat{F}_m^+ + \left(m+\frac{1}{2}\right)\hat{F}_{m-1}^+, \tag{14.19}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{F}_m^-(t) = -\left(m+\frac{1}{2}\right)\hat{F}_m^- + \sqrt{m(m+1)}\hat{F}_{m-1}^-, \tag{14.20}$$

for

$$\hat{F}_m^{\pm} = |\psi_m^{(+)}\rangle\langle\psi_m^{(+)}| \pm |\psi_m^{(-)}\rangle\langle\psi_m^{(-)}|. \tag{14.21}$$

The equations (14.19) and (14.20) are solved easily by following the Sect. 10.3. The resulting expressions for $\hat{F}_m^{\pm}(t)$, which may also be verified by direct substitution in the respective equations, read

$$\hat{F}_m^+(t) = \exp\left(-(2m+1)\kappa t\right) \times \sum_{n=0}^{m} \frac{\Gamma\left(m+3/2\right)\left(\exp(2\kappa t)-1\right)^{m-n}}{(m-n)!\,\Gamma\left(n+3/2\right)}\hat{F}_n^+(0), \tag{14.22}$$

and

$$\hat{F}_m^-(t) = \exp\left(-(2m+1)\kappa t\right) \times \sum_{n=0}^{m} \sqrt{\frac{m+1}{n+1}}\frac{m!\left(\mathrm{e}^{2\kappa t}-1\right)^{m-n}}{(m-n)!\,n!}\hat{F}_n^-(0). \tag{14.23}$$

The solution of the equations above, on substitution in (14.9) provides the description of the system in the secular approximation. For some detailed numerical results see [103]. For a different approach to the strong-coupling limit, see [128].

***N* Two-Level Atoms.** In order to apply the secular approximation in this case, we need to know the eigenvalues and the eigenstates of $\hat{H}_0$ for a general value of N. As stated in Sect. 11.2, the eigenvalue problem of $\hat{H}_0$ can be solved analytically only if the eigenvalue M of $\hat{a}^\dagger\hat{a} + \hat{S}_z$ is small. We let N be arbitrary and $M = -N/2, -N/2+1, -N/2+2$. We also let $\omega = \omega_0$.

The state $|0, -N/2\rangle$ is the only eigenstate of $\hat{H}$ if $M = -N/2$. It corresponds to the eigenvalue $-N\hbar\omega/2$.

For $M = -N/2+1$ there are two eigenstates of $\hat{H}$:

$$|\psi_0^\pm\rangle = \frac{1}{\sqrt{2}}\left[|0, -N/2+1\rangle \pm |1, -N/2\rangle\right]. \tag{14.24}$$

The corresponding eigenvalues of $\hat{H}$ are $\hbar(-N\omega/2 + \mu_N^\pm)$,

$$\mu_N^\pm = \omega \pm g\sqrt{N}. \tag{14.25}$$

Thus, if the initial excitation is $-N/2+1$ then the N-atom system behaves mathematically like a two-level one.

Next, for $M = -N/2+2$, $\hat{H}$ admits three eigenstates:

$$|\psi_i\rangle = \sum_{j=1}^{3} a_{ij}|j-1, \ -\frac{N}{2}+3-j\rangle, \tag{14.26}$$

where a_{ij} are the elements of the matrix $\hat{A}$ defined by

$$\hat{A} = \begin{pmatrix} \sqrt{(N-1)/2(2N-1)} & 1/\sqrt{2} & \sqrt{N/2(2N-1)} \\ -\sqrt{N/2N-1} & 0 & \sqrt{(N-1)/2N-1} \\ \sqrt{(N-1)/2(2N-1)} & -1/\sqrt{2} & \sqrt{N/2(2N-1)} \end{pmatrix} \tag{14.27}$$

The eigenstates (14.26) correspond to the eigenvalues $\hbar(-N\omega/2 + \mu_i)$ $(i = 1, 2, 3)$ where

$$\mu_{1,3} = 2\omega_0 \pm g\sqrt{4N-2}, \quad \mu_2 = 2\omega_0. \tag{14.28}$$

The inverse of the relation (14.26) is

$$|j-1, \ -\frac{N}{2}+3-j\rangle = \sum_{i=1}^{3} a_{ij}|\psi_i\rangle. \tag{14.29}$$

If the initial excitation of the system is $M \leq -N/2+2$ then it is confined to the subspace formed by the eigenstates constructed above. We use these results in the next section for determining the atomic response to an external field.

14.3 Response to an External Field

A useful means of probing the characteristic features of atom-cavity interaction is by monitoring its response to a weak external drive. It is determined

by the method outlined in Chap. 9. Recall that it consists in evaluating an appropriate susceptibility of the system. We use that approach to examine the response of a system of two-level atoms in a cavity to (i) a monochromatic field, and (ii) a bichromatic field.

14.3.1 Linear Response to a Monochromatic Field

We let the atoms to be driven by a monochromatic field of frequency ν. Recall from Chap. 9 that the rate of absorption of the energy by the system from the field is given by (9.26):

$$W_{\rm a}(\nu) \sim {\rm Re}\left[{\rm Tr}\left\{\hat{S}_-\left(\hat{\hat{L}}+{\rm i}\nu\right)^{-1}\left[\hat{S}_+,\ \hat{\rho}_{\rm ss}\right]\right\}\right]$$
$$= \sqrt{N}{\rm ReTr}\left[\hat{S}_-\left(\hat{\hat{L}}+{\rm i}\nu\right)^{-1}|0,-N/2+1\rangle\langle 0,-N/2|\right]\Bigg], \quad (14.30)$$

where we have used the fact that $\hat{\rho}_{\rm ss}$ is given by (14.7)). The expression (14.30) can, of course, be evaluated exactly by using the results of the last section. However, simpler expression is obtained in the secular approximation. To apply it, we invoke (14.24) to express $|0,-N/2+1\rangle$ in terms of the dressed states. Then, following the procedure outlined in the last section, it is straightforward to see that, in the secular approximation,

$$\left(\hat{\hat{L}}+{\rm i}\nu\right)|\psi_0^{\pm}\rangle\langle 0,-\frac{N}{2}| = A_{\pm}|\psi_0^{\pm}\rangle\langle 0-N/2| \quad (14.31)$$

$$A_{\pm} = \left[-{\rm i}\left(\omega_0-\nu\pm g\sqrt{N}\right)-\frac{\kappa+\gamma N}{2}\right]. \quad (14.32)$$

The equation (14.31) yields

$$\left(\hat{\hat{L}}+{\rm i}\nu\right)^{-1}|0,-\frac{N}{2}+1\rangle\langle 0,-\frac{N}{2}|$$
$$= \frac{1}{\sqrt{2}}\left[A_+^{-1}|\psi_0^+\rangle + A_-^{-1}|\psi_0^-\rangle\right]\langle 0,-\frac{N}{2}|$$
$$= \frac{1}{2}\left[\{A_+^{-1}+A^{-1}\}|0,-N/2+1\rangle + \{A_+^{-1}-A_-^{-1}\}|1,-N/2\rangle\right]\langle 0,-N/2|. \quad (14.33)$$

On substituting this in (14.30) it follows that the rate of absorption of energy from the applied field is proportional to

$$W_{\rm a}(\nu) \sim \left[\left(\omega_0-\nu-g\sqrt{N}\right)^2+\left(\frac{\kappa+\gamma N}{2}\right)^2\right]^{-1}$$
$$+\left[\left(\omega_0-\nu+g\sqrt{N}\right)^2+\left(\frac{\kappa+\gamma N}{2}\right)^2\right]^{-1}. \quad (14.34)$$

The rate of absorption thus exhibits resonances at $\omega_0 \pm g\sqrt{N}$ as a function of the applied frequency ν. Compare it with the case of an atom in free space. In that case, the absorption exhibits resonance at the atomic transition frequency ω_0. The splitting of the absorption resonance at ω_0 in a damped cavity to two resonances at $\omega_0 \pm g\sqrt{N}$ is referred to as the *vacuum field Rabi splitting*. The reader may verify by numerical evaluation that the exact expression for $W_{\rm a}(\nu)$ for $\sqrt{N}|g| \gg \kappa, N\gamma$ is in good agreement with the one derived above in the secular approximation.

14.3.2 Nonlinear Response to a Bichromatic Field

Next, we examine the nonlinear response of a two-level atom coupled strongly with a single cavity mode to a bichromatic field consisting of frequencies ω_1, ω_2. This coupling is described by the hamiltonian given by (12.42). We are interested in the response at the frequency $\Omega = 2\omega_1 - \omega_2$. It is characterized by the susceptibility given by (12.43) read with (12.44). The superoperator $\hat{\hat{L}}$ in this case is given by (14.3) plus the $\hat{\hat{L}}_c$ where $\hat{\hat{L}}_c$, defined in (8.106), incorporates effects of collisions that the atom in question undergoes with other atoms. We assume $\bar{n} = 0$ so that $\hat{\rho}_{\rm ss}$ is given by (14.7). Recall that in Chap. 12 we investigated this process in free space.

We evaluate $\chi^{(3)}(\omega_1, \omega_1, -\omega_2)$ numerically and compute the signal

$$S(\Omega) = |\chi^{(3)}(\omega_1, \omega_1, -\omega_2)| \tag{14.35}$$

as a function of ω_2. We assume that $\omega_1 = \omega_0$. The resonances in the susceptibility are expected to occur at the imaginary parts of the eigenvalues of $\hat{\hat{L}}$. In the absence of damping, $\hat{\hat{L}} \to \hat{\hat{L}}_{\rm a-f}$. The eigenvalues of $\hat{\hat{L}}_{\rm a-f}$ are purely imaginary and are given by $\mathrm{i}\{\mu_m - \mu_n\}$ where $\{\mu_m\}$ are the eigenvalues of $\hat{H}_{\rm a-f}$. We observe from the expression for the susceptibility that if $\bar{n} = 0$ then it involves trasnsitions between the states up to the second excited manifold of the dressed states. The corresponding eigenvalue spectrum is depicted in in Fig. 14.1. The figure shows also the allowed transitions between the levels. The eigenvalues are expected to be only slightly perturbed if the damping is small. The resonances in the susceptibility are then expected to occur at

(a) $\omega_2 = \omega_0 \pm g$,

(b) $\omega_2 = 2\omega_1 - \omega_0 \pm g$,

(c) $\omega_2 = 2\omega_1 - \omega_0 \pm g\left(\sqrt{2} \pm 1\right)$.

Like we found in the case of a free atom, the collisional damping may lead to the appearance of new resonances. In Fig. 14.2, we plot normalized signal $S_m = S(\Omega)/S_{\max}(\Omega)$ as a function of $(\omega_2 - \omega_0)/g$ for $\kappa = 0.03g$ and $\gamma = 0.01g$. The solid line curve is in the absence of atomic collisions ($\gamma_c = 0$) whereas the dashed curve is for $\gamma_c = 0.04$. Note that in the absence of collisions, the

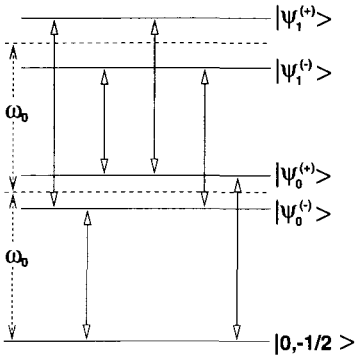

Fig. 14.1. Schematic diagram of the spectrum of the Jayens-Cummings hamiltonian of a single two-level atom up to the second excited manifold.

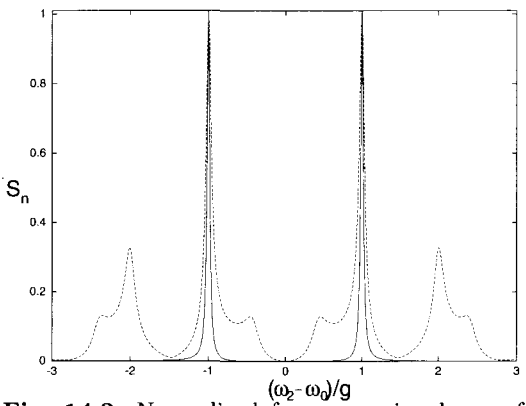

Fig. 14.2. Normalized four-wave signal as a function of $\delta_2 = (\omega_2 - \omega_0)/g$ for $\kappa = 0.03g$, $\gamma = 0.01g$ $\gamma_c = 0$ (*solid line curve*) and $\gamma_c = 0.04g$ (*dashed curve*).

signal exhibits resonances only at $\omega_0 \pm g$ (resonance (a) above). Since ω_1 is assumed to be the equal to ω_0, the resonance (b) is same as (a). However, the resonance (c) is different but is suppressed by the one at $\omega_0 \pm g$. The dashed curve for the case when the atom undergoes collisions exhibits the resonances (c). It also exhibits the collision induced Bloembergen resonances at $\omega_2 - \omega_0 = \pm 2g$. These do not correspond to any dipole allowed transition.

The resonance (a) is due to transitions to the first manifold of the spectrum of the interacting atom-field system. This part of the spectrum is the same as the one for a harmonic oscillator model in a classical field. The part of the spectrum corresponding to the second and higher manifolds is characteristic of a two-level model and the quantized field. This part is brought out by the resonance (c). These resonances are thus a signature of field quantization and hence may be referred to as *quantum resonances*. The observations here establish important role played by collisions in bringing out the quantum resonances. For further details, see [129]. That reference also shows that the

results of the secular approximation in this instance are not only in quantitative but also in qualitative disagreement with exact numerical results. The secular approximation need not hold for evaluation of a nonlinear susceptibility. The second of the reference [129] discusses also the case of mixing by a collective system of N atoms. For a description of quantum effects in four-wave mixing by a trapped atom, see [130].

14.4 The Micromaser

A setup in cavity QED that offers immense possibilities of realizing a variety of quantum effects is the micromaser [131]-[135]. A micromaser refers to a setup in which atoms prepared in a particular state pass through the cavity one at a time. In these experiments, the field is too weak for measurements. The experimentally measurable property in these experiments is the atomic population. The statistical characteristics of the cavity field are, therefore, required to be determined in terms of the statistics of the atomic population in a given state. The relationship between the statistical properties of the cavity field and that of the atomic population is, however, still an open question even for the simple model of a two-level atom in a single-mode cavity. In the following we outline briefly the theory of a micromaser.

14.4.1 Density Operator of the Field

Let an atom, prepared in a desired state (henceforth referred to as an active state) described by the density operator $\hat{\rho}_{\rm a}$ pass one at a time through a cavity. Let $\hat{\rho}_{\rm f}(t_i)$ be the density operator of the field at the time the $\mathrm{i}^{\rm th}$ atom enters the cavity. Let $t_{\rm int}$ be the time taken by an atom to traverse the cavity. If $\hat{\hat{L}}_{\rm int}$ is the Liouvillean describing the evolution of the combined system of the atom and the cavity field then the state of the cavity field at the time $t_i + t_{\rm int}$ when the ith atom exits the cavity is evidently given by

$$\begin{aligned}\hat{\rho}_{\rm f}(t_i + t_{\rm int}) &= {\rm Tr}_{\rm atom}\left[\exp\left(\hat{\hat{L}}_{\rm int} t_{\rm int}\right)\hat{\rho}_{\rm a}\hat{\rho}_{\rm f}(t_i)\right] \\ &\equiv \hat{\hat{F}}(t_{\rm int})\rho_{\rm f}(t_i), \end{aligned} \tag{14.36}$$

Where $\hat{\hat{F}}(t_{\rm int})$ is the so called *production* Liouvillean:

$$\hat{\hat{F}}(t_{\rm int}) = {\rm Tr}_{\rm atom}\left[\exp\left(\hat{\hat{L}}_{\rm int} t_{\rm int}\right)\hat{\rho}_{\rm a}\right]. \tag{14.37}$$

Let the field evolve under the action of the Liouvillean $\hat{\hat{L}}_{\rm f}$ from the time $t_i + t_{\rm int}$ the ith atom exits the cavity till the time t_{i+1} of the entry of the $(i+1)$th atom. The state of the cavity field at that time is given by

$$\hat{\rho}_{\rm f}(t_{i+1}) = \exp\left(\hat{\hat{L}}_{\rm f}(t_{i+1}-t_i-t_{\rm int})\right) {\rm Tr}_{\rm atom}\left[\exp\left(\hat{\hat{L}}_{\rm int}t_{\rm int}\right)\hat{\rho}_{\rm a}\hat{\rho}_{\rm f}(t_i)\right]$$
$$\approx \exp\left(\hat{\hat{L}}_{\rm f}\tau\right) {\rm Tr}_{\rm atom}\left[\exp\left(\hat{\hat{L}}_{\rm int}t_{\rm int}\right)\hat{\rho}_{\rm a}\hat{\rho}_{\rm f}(t_i)\right]$$
$$\equiv \hat{\hat{D}}(\tau)\hat{\hat{F}}(t_{\rm int})\hat{\rho}_{\rm f}(t_i),\ \tau = t_{i+1}-t_i, \tag{14.38}$$

where

$$\hat{\hat{D}}(\tau) = \exp\left(\hat{\hat{L}}_{\rm f}\tau\right) \tag{14.39}$$

is the *damping* Liouvillean. The second line in (14.38) is obtained by assuming, in accordance with the usually encountered experimental conditions, that the time interval $\tau = t_{i+1} - t_i$ between the arrival of two successive atoms in the cavity is very much longer than the time $t_{\rm int}$ that an atom takes to traverse the cavity. Note that the trace of the density operator does not change during an evolution. Hence,

$${\rm Tr}[\hat{\hat{D}}\hat{\rho}] = {\rm Tr}[\hat{\rho}], \qquad {\rm Tr}[\hat{\hat{F}}\hat{\rho}] = {\rm Tr}[\hat{\rho}] \tag{14.40}$$

for any $\hat{\rho}$.

The equation (14.38) determines the dynamics of the field in a micromaser for fixed values of τ and $t_{\rm int}$. Its formal solution reads

$$\hat{\rho}_{\rm f}(t_n) = \left[\hat{\hat{D}}(\tau)\hat{\hat{F}}(t_{\rm int})\right]^n \hat{\rho}_{\rm f}(0). \tag{14.41}$$

This may be evaluated, for example, by the method of expansion of $\hat{\rho}(0)$ in terms of the eigenvectors of $\hat{\hat{D}}(\tau)\hat{\hat{F}}(t_{\rm int})$. Let $\hat{\rho}_\nu$ be an eigenvector of $\hat{\hat{D}}\hat{\hat{F}}$ corresponding to the eigenvalue μ_ν, i.e.

$$\hat{\hat{D}}\hat{\hat{F}}\hat{\rho}_\nu = \mu_\nu\hat{\rho}_\nu. \tag{14.42}$$

Let $\hat{\tilde{\rho}}_\nu$ be the eigenvector of the adjoint of $\hat{\hat{D}}\hat{\hat{F}}$ corresponding to the eigenvalue μ_ν^*. Let us represent an operator $\hat{f}$ by $|f)$. The expression (14.41) may then be written as

$$|\rho_{\rm f}(t_n)) = \sum_\nu \exp\left[n\ln(\mu_\nu)\right]|\rho_\nu)\,(\tilde{\rho}_\nu|\rho_{\rm f}(0))\,, \tag{14.43}$$

the $(A|B)$ denotes scalar product of $\hat{A}$ and $\hat{B}$. We note that

1. In the problems of interest, one of the eigenvalues of $\hat{\hat{D}}\hat{\hat{F}}$ is unity whereas the magnitude of all its other eigenvalues is less than one. Note that the terms in the summation in (14.43) for which $|\mu_\nu| < 1$ vanish in the limit $n \to \infty$. Hence, as $n \to \infty$, $\hat{\rho}_{\rm f}(t_n) \to \hat{\rho}_{\rm ss}$ where $\rho_{\rm ss}$ is the eigenstate of $\hat{\hat{D}}(\tau)\hat{\hat{F}}(t_{\rm int})$ corresponding to eigenvalue unity i.e.

$$\hat{\hat{D}}(\tau)\hat{\hat{F}}_0(t_{\rm int})\rho_{\rm ss} = \rho_{\rm ss}. \tag{14.44}$$

2. The time scale of approach to the steady state is determined by the eigenvalue nearest to unity in magnitude. Thus, if μ_1 is the eigenvalue nearest to unity in magnitude then the steady state will be reached when a number $n \sim 1/\ln(1/|\mu_1|)$ of atoms have passed through the cavity. If R is the rate of pumping then this corresponds to the time scale $t \sim 1/R\ln(1/|\mu_1|)$.

The relation (14.38) determines the state of the field when the time interval τ between the arrival of two successive atoms in to the cavity and the time of interaction $t_{\rm int}$ are constants. These times, however, need not be constants. Consider first the time τ. The spread in τ arises due to the fact that the process of preparing atoms in the desired state at the entry of the cavity is generally a random process. For, in the experiments, the atoms are pumped in to the cavity from a source at regular intervals of time, say, T. At the cavity, the atoms are excited to the active state by shining appropriate lasers. The process of excitation is random. It is described adequately by assuming that the atoms are excited to the active state with probability p and that there is no correlation between the excitation of any two atoms. If K atoms arrive at the cavity in time $t_K = KT$ then the probability $P(N, K)$ that N of them are excited to the active state is given, under the aforementioned assumptions, by the Binomial distribution

$$P(N,K) = \binom{K}{N} p^N (1-p)^{K-N}. \tag{14.45}$$

Now, let $\rho_{\rm f}(N, t_K)$ denote the state of the field at the time t_K of the arrival of the $K^{\rm th}$ atom under the condition that N atoms before it got excited to the active state. At the cavity, it may get excited to the active state with probability p or not with probability $1-p$. If it is not excited then it passes through the cavity leaving the state of the field unchanged. Else it interacts with the cavity field transforming its state, by virtue of (14.36), by the action of $\hat{\hat{F}}$ on it. In either case, the field decays for time T till the arrival of the next atom numbered $K+1$. The density operator $\rho_{\rm f}(N, t_{K+1})$ of the field seen by atom number $K+1$ is thus expressible as

$$\hat{\rho}_{\rm f}(N, t_{K+1}) = \exp\left(\hat{\hat{L}}_{\rm f} T\right) \Big[(1-p)\hat{\rho}_{\rm f}(N, t_K) + p\hat{\hat{F}}\hat{\rho}_{\rm f}(N-1, t_K)\Big]. \tag{14.46}$$

This recurrence relation may be used to determine the field density operator $\hat{\rho}_{\rm f}(t_K)$ at the time t_K with no regard for the number of active atoms traversing the cavity or the field density operator $\hat{\rho}_{\rm f}(N)$ at the time of entry of the $N^{\rm th}$ active atom without any reference to how long it takes to prepare that many active atoms.

The density operator $\hat{\rho}_{\rm f}(t_K)$ at the time t_K is given by $\hat{\rho}_{\rm f}(N, t_K)$ summed over all N. On carrying summation over N in (14.46) it follows that

$$\hat{\rho}_{\rm f}(t_{K+1}) = \exp\left(\hat{\hat{L}}_{\rm f} T\right)\left[(1-p) + p\hat{\hat{F}}\right]\hat{\rho}_{\rm f}(t_K). \tag{14.47}$$

The density operator $\hat{\rho}_{\rm f}(N)$, on the other hand, is given by $\hat{\rho}_{\rm f}(N, t_K)$ summed over all K:

$$\hat{\rho}_{\rm f}(N+1) = \frac{p\exp\left(\hat{\hat{L}}_{\rm f} T\right)\hat{\hat{F}}}{1-(1-p)\exp\left(\hat{\hat{L}}_{\rm f} T\right)}\hat{\rho}_{\rm f}(N). \tag{14.48}$$

Let us examine the steady state solution of the maps (14.47) and (14.48). The steady state of the map (14.47) is the state reached in the asymptotic limit $t_K \to \infty$ whereas that of the map (14.48) is the state reached in the limit $N \to \infty$. Those states correspond to the solutions $\hat{\rho}_{\rm f}(t_{K+1}) = \hat{\rho}_{\rm f}(t_K)$, $\hat{\rho}_{\rm f}(N+1) = \hat{\rho}_{\rm f}(N)$ respectively of (14.47) and (14.48). The steady state $\hat{\rho}_{\rm ss}$ of (14.47) is the solution of the operator equation

$$\left[1 - \exp\left(\hat{\hat{L}}_{\rm f} T\right)\left((1-p) + p\hat{\hat{F}}\right)\right]\hat{\rho}_{\rm ss} = 0. \tag{14.49}$$

It is easy to see that the steady state of the map (14.48) also obeys (14.49). Hence the steady state density operator of the cavity field is the same whether the dynamics is described in terms of time or in terms of the number of active atoms traversing the cavity.

An experimentally important case of random pumping is the Poisson pumping. This corresponds to the limit $p \to 0$, $T \to 0$ of binomial pumping such that $p/T \to R$ where the constant R is the average rate of pumping of the active atoms. In this limit, the atoms enter the cavity continuously. In other words, in the Poisson limit, the discrete time t_K becomes a continuous variable t. Hence we can write $\hat{\rho}_{\rm f}(N, t_{K+1}) - \hat{\rho}_{\rm f}(N, t_K) = T(\mathrm{d}/\mathrm{d}t)\hat{\rho}_{\rm f}(N, t)$. The equation (14.47) then reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}_{\rm f}(t) = \left[\hat{\hat{L}}_{\rm f} + R\left(\hat{\hat{F}} - 1\right)\right]\hat{\rho}_{\rm f}(t). \tag{14.50}$$

Finally, the density operator of the field after the passage of a fixed number N of active atoms, irrespective of the time for the case of Poissonian pumping, is found from (14.48) to be given by

$$\hat{\rho}_{\rm f}(N+1) = \left(1 - \hat{\hat{L}}_{\rm f}/R\right)^{-1}\hat{\hat{F}}\hat{\rho}_{\rm f}(N). \tag{14.51}$$

The properties of the cavity field including the effect of the pumping statistics may thus be studied for any model of atom-field interaction.

We must also account for the fact that all the atoms may not move with the same velocity. The spread in the atomic velocities results in a spread in the time of their interaction $t_{\rm int}$. This may be included by averaging the density operator found above over the distribution $W(t_{\rm int})$ of $t_{\rm int}$. In realistic situations the atomic velocities follow the Maxwellian distribution. The average over $W(t_{\rm int})$ may be carried knowing the functional dependence of operators on $t_{\rm int}$. It has been computed in [132] for the two-level atomic micromaser with a single mode cavity discussed next.

14.4.2 Two-Level Atomic Micromaser

Consider two-level atoms pumped in to a cavity. The atoms interact resonantly with a single cavity mode by executing one-photon transitions between its levels $|g\rangle$ and $|e\rangle$. This interaction is described by (14.1) with $\omega = \omega_0$. The micromaser theory is based on determining the field superoperators defined in (14.37) and (14.39). Assume the Q-factor of the cavity to be very high so that the field damping is negligibly small during the time of transit of the atom. The evolution of the combined system of the atom and the field is then governed only by (14.1) so that

$$\exp\left[\hat{\hat{L}}_{\rm int} t_{\rm int}\right] \hat{\rho}_{\rm a}\hat{\rho}_{\rm f} = \exp\left[-{\rm i}\hat{H} t_{\rm int}\right] \hat{\rho}_{\rm a}\hat{\rho}_{\rm f} \exp\left[{\rm i}\hat{H} t_{\rm int}\right]. \tag{14.52}$$

The expression for the production operator may be found by using the results of Chap. 11. Show that if the atoms enter the cavity in their excited state then

$$\hat{\hat{F}}\hat{\rho}_{\rm f} = \cos\left(g t_{\rm int}\hat{\lambda}\right)\hat{\rho}_{\rm f}\cos\left(g t_{\rm int}\hat{\lambda}\right) + \hat{a}^{\dagger}\frac{\sin\left(g t_{\rm int}\hat{\lambda}\right)}{\hat{\lambda}}\hat{\rho}_{\rm f}\frac{\sin\left(g t_{\rm int}\hat{\lambda}\right)}{\hat{\lambda}}\hat{a}, \tag{14.53}$$

$$\hat{\lambda} = g\sqrt{\hat{a}^{\dagger}\hat{a}+1} \tag{14.54}$$

We examine the characteristics of the micromaser first by ignoring the field damping at all times.

Undamped Micromaser. A simplified version of the micromaser is obtained by assuming that the field does not decay significantly even during the time interval between the arrival of two atoms in to the cavity [136]. In this case, $\hat{\hat{L}}_{\rm f}\hat{\rho} = -{\rm i}\omega[\hat{a}^{\dagger}\hat{a},\ \hat{\rho}]$. Use this form of $\hat{\hat{L}}_{\rm f}$ and (14.53) for $\hat{\hat{F}}$ in (14.38), take the matrix element in the number state $|m\rangle$ of the field to obtain (with $\rho_{mn} \equiv \langle m|\hat{\rho}_{\rm f}|n\rangle$)

$$\langle m|\hat{\rho}_{\rm f}(k+1)|n\rangle \equiv \rho_{mn}(k+1) = \alpha_{m+1}\alpha_{n+1}\rho_{mn}(k) + \beta_m\beta_n\rho_{m-1n-1}(k), \tag{14.55}$$

$$\alpha_m = \cos(g t_{\rm int}\sqrt{m}), \qquad \beta_m = \sin(g t_{\rm int}\sqrt{m}). \tag{14.56}$$

The equation (14.55) gives the density operator of the field at the time of entry of $(k+1)$th atom in terms of that at the time of entry of kth atom in to the cavity. Let us examine the steady state solution of this equation. Consider first the diagonal elements. Writing $p_m = \rho_{mm}$ we see that

$$p_m(k+1) = \alpha_{m+1}^2 p_m(k) + \beta_m^2 p_{m-1}(k). \tag{14.57}$$

Its steady state solves

$$\beta_{m+1}^2 p_m = \beta_m^2 \rho_{m-1} \tag{14.58}$$

This shows that $\beta_1^2 p_0 = \beta_2^2 p_1 = \cdots = \beta_{n+1}^2 p_n = \cdots = 0$. Hence, a non-trivial steady-state is possible only if

$$\beta_{N_1+1} = 0 \tag{14.59}$$

for some positive integer N_1, which is such that, for an integer p,

$$g t_{\rm int} \sqrt{N_1 + 1} = p\pi. \tag{14.60}$$

In this case $\rho_{N_1 N_1} \neq 0$ whereas $\rho_{mm} = 0$ for $m < N_1$. Note that if (14.60) holds then (14.59) is satisfied for all those positive integers N_i which satisfy

$$N_i = q_i^2 (N_1 + 1) - 1, \qquad q_1 = 1. \tag{14.61}$$

Hence, under the *trapping condition* (14.60)

$$p_m = \sum_i C_i \delta_{m,N_i}. \tag{14.62}$$

In order to determine C_i, note from (14.57) that under the condition (14.60), the equations for $p_m(k+1)$ for the values $m \geq N_1+1$ are decoupled from $p_n(k)$ $n \leq N_1$. It may in fact be verified that the equations reduce to uncoupled blocks defined by $N_{i-1}+1 \leq m, n \leq N_i$ (with $N_0 = -1$). Since the equations in each block are decoupled from each other, it implies that the trace of $\hat{\rho}$ in a block remains unchanged during the evolution. Hence

$$C_i = \sum_{m=N_{i-1}+1}^{N_i} p_m(0), \tag{14.63}$$

$p_m(0)$ being the initial photon number distribution function. Thus, if the cavity is initially in the vacuum state then $\rho_{mm}(0) = \delta_{m0}$ and $C_i = \delta_{i1}$ i.e. the cavity field is the number state $|N_1\rangle$. If the initial field has components in other blocks then the steady state would be a mixture of the number states.

For the steady state of an undamped micromaser when the atoms enter the cavity in a superposition state, see [137]. A micromaser operating with three-level atoms in a two-mode cavity has been analyzed in [138]. A micromaser in which atomic levels are coupled by two channels is studied in [139].

Damped Micromaser. Let us now account for the field damping during the time interval between the arrival of successive atoms. Let the field damping be described by $\hat{\Lambda}_{\rm f}$ defined in (14.5) so that $\hat{\hat{L}}_{\rm f} \equiv \hat{\Lambda}_{\rm f}$. Let $\bar{n} = 0$. We substitute that form of $\hat{\hat{L}}_{\rm f}$ in (14.39) to evaluate the field damping superoperator $\hat{\hat{D}}(\tau)$. Its matrix elements in the photon number representation are given by (13.34). It is then straightforward to show that, owing to the map (14.38), and (14.57), the photon number distribution p_m obeys

$$\begin{aligned} p_m(t_{i+1}) &= \sum_{k=m} D_{mk} \langle k | \hat{\hat{F}}(t_{\rm int}) \hat{\rho}_{\rm f}(t_i) | k \rangle \\ &= \sum_{k=m}^{\infty} D_{mk} \left[\alpha_{k+1}^2 p_k(t_i) + \beta_k^2 p_{k-1}(t_i) \right], \end{aligned} \tag{14.64}$$

$$D_{mk} = \frac{k!}{(k-m)!m!} d^m (1-d)^{k-m}, \qquad d = \exp(-2\kappa\tau). \tag{14.65}$$

The equation (14.64) shows that if there is a trap at N_k then $p_{N_k+n}(t_{i+1})$ $(n \geq 1)$ are no longer coupled with $p_m(t_i)$ $(m \leq N_k)$. The equation for $p_m(k+1)$ in a block is coupled with $p_m(k)$ in higher blocks but not with the ones in lower blocks. Hence, if $p_m(t_0) = 0$ for m in blocks higher than the M^{th} then $p_m(t_i) = 0$ for all t_i for m in a block higher than the M^{th}.

An interesting and useful consequence of the trapping condition is that p_m in the steady state is confined to the lowest block. In order to see that, perform the summation over m from zero to N on the steady state form of (14.64). Owing to (A.14), $D_{mk} = 0$ if $k < m$. Hence, we may extend the lower limit of summation over k in (14.64) to zero. The summation over m then involves only D_{mk}. Break the summation over k in two parts, one running from zero to N and another from $N+1$ to ∞ to obtain

$$\sum_{m=0}^{N} p_m = \sum_{k=0}^{N} F_k \sum_{m=0}^{N} D_{mk} + \sum_{k=N+1}^{\infty} F_k \sum_{m=0}^{N} D_{mk}, \tag{14.66}$$

the F_k being the m-independent terms under the summation in (14.64). Since the value of k in the first sum above is less than the upper limit N of the summation over m in it, and since $D_{mk} = 0$ for $m > k$, the upper limit of summation over m there may be replaced by k. The resulting binomial sum over m then yields unity. Change the sumation index over $k \to k+1$ in the term having β_k to get

$$-\beta_{N+1}^2 p_N + \sum_{k=N+1}^{\infty} F_k \sum_{m=0}^{N} D_{mk} = 0. \tag{14.67}$$

We let $N = N_1$, which is the lowest value of N for which $\beta_{N+1} = 0$, to arrive at

$$\sum_{k=N_1+1}^{\infty} \left(\alpha_{k+1}^2 \tilde{D}_{N_1 k} + \beta_{k+1}^2 \tilde{D}_{N_1 k+1} \right) p_k = 0, \tag{14.68}$$

where

$$\tilde{D}_{Nk} = \sum_{m=0}^{N} D_{mk} \tag{14.69}$$

are positive definite if $d \neq 1$. The term in the parentheses in (14.68) is, therefore, positive definite if $d \neq 1$. Since p_k are positive semidefinite, it follows that in order to satisfy (14.68) they should vanish if $k \geq N_1 + 1$. In case of $d = 1$, i.e. if the cavity damping is ignored, then $\tilde{D}_{N_1 k} = 0$ for $k > N_1$. The equation (14.68) then can be satisfied for non-zero p_k. Thus, the problem of finding the steady state reduces to solving equations only in the first block. This is evidently a significant simplification.

The problem of determining the time-scale of approach to equilibrium, however, requires an elaborate analysis of the eigenvalues. We refer to [140] for the details of such an analysis. Here we quote the substance of the results. Those results state that if $\epsilon \equiv \kappa\tau \ll 1$ then the population in the k^{th} block leaks to the lower block on the time scale $\sim \epsilon^{N_k - N_{k-1}}$. The population in the lowest block approaches the steady state on the time scale of the cavity damping time κ^{-1}. For a numerical study of the time-scale, see [141].

Approximation Method. Since solving even the zero temperature micromaser photon number distribution equation (14.64) is generally a formidable task, approximation methods are employed to investigate its characteristics. These methods consist in reducing the stroboscopic map (14.38) to a continuous-time evolution by noting that $[\hat{\rho}_{\text{f}}(t_{i+1}) - \hat{\rho}_{\text{f}}(t_i)]/(t_{i+1} - t_i) \approx (\text{d}/\text{d}t_i)\hat{\rho}_{\text{f}}(t_i)$ if $t_{i+1} - t_i \equiv \tau \ll t_i$. On replacing t_i by t, (14.38) in this approximation reduces to ($\tau = R^{-1}$)

$$\frac{\text{d}}{\text{d}t}\hat{\rho}_{\text{f}}(t) = R \ln\left[\hat{\hat{D}}(\tau)\hat{\hat{F}}(t_{\text{int}})\right]\hat{\rho}_{\text{f}}(t). \tag{14.70}$$

This equation is further approximated by treating the logarithm of operators as c-number logarithms to obtain

$$\frac{\text{d}}{\text{d}t}\hat{\rho}_{\text{f}}(t) = \left[\hat{\hat{L}}_{\text{f}} + R\ln(\hat{\hat{F}}(t_{\text{int}}))\right]\hat{\rho}_{\text{f}}(t). \tag{14.71}$$

Furthermore, $\ln(\hat{\hat{F}}(t_{\text{int}}))$ is expressed in powers of $1 - \hat{\hat{F}}(t_{\text{int}})$ retaining terms to a desired order.

In the case of an atom in a single-mode field, further approximation is introduced to treat the photon number m as a continuous variable μ. This approximation applied to the Jaynes-Cummings model described to second order in $1 - \hat{\hat{F}}(t_{\text{int}})$ in (14.71) then turns out to be a Fokker-Planck equation. We refer the reader to [140, 142] for comparison of the exact and approximate results.

14.4.3 Atomic Statistics

As mentioned before, the characteristics of a micromaser are examined experimentally by studying the statistics of the atomic population and correlation between the population of atoms exiting the cavity at different times. The analytical results for the variance in the observed population of atoms exiting the cavity and correlation between atomic population at different times were reported first in [143] by assuming the pumping mechanism to be Poissonian. It included also the effects of finite detector efficiency. See also [144, 145] for analysis of atomic correlations. The study of the atomic statistics has been carried in [146] by constructing the joint distribution for observing atoms exiting the cavity at different times in one state or the other. It is applicable to a binomial pumping which includes Poissonian pumping as a special case. See [147, 148] for experimental results.

Appendices

A. Some Mathematical Formulae

- The delta function is defined by

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k \exp(\mathrm{i}kx). \tag{A.1}$$

$$\delta(\xi)\delta(\xi^*) = \frac{1}{\pi^2}\int \exp\{\mathrm{i}(z\xi + z^*\xi^*)\}\,\mathrm{d}^2 z \tag{A.2}$$

where the integration is over the complex plane.

$$\int_0^{\infty} \mathrm{d}k \exp(\pm \mathrm{i}kx) = \pi\delta(x) \pm \mathrm{i}P\frac{1}{x} \tag{A.3}$$

where P indicates Cauchy principal value defined by

$$P\int \frac{\mathrm{d}x}{x}F(x) = \lim_{\epsilon\to 0}\left[\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty}\right]\mathrm{d}x\ \frac{1}{x}F(x). \tag{A.4}$$

$$\lim_{\epsilon\to 0}\frac{1}{x \pm \mathrm{i}\epsilon} = P\frac{1}{x} \mp \mathrm{i}\pi\delta(x). \tag{A.5}$$

$$\delta(x-y)F(x) = \delta(x-y)F(y), \tag{A.6}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\delta(x-y) = -\frac{\mathrm{d}}{\mathrm{d}y}\delta(x-y) \tag{A.7}$$

$$\frac{1}{2}\lim_{\gamma\to\infty}\gamma\exp(-\gamma|\tau|) = \delta(\tau). \tag{A.8}$$

$$\lim_{a\to 0}\frac{1}{a\sqrt{\pi}}\exp(-x^2/a^2) = \delta(x). \tag{A.9}$$

- If

$$\tilde{F}(\xi,\xi^*) = \frac{1}{\pi^2}\int F(z,z^*)\exp\{\mathrm{i}(z\xi + z^*\xi^*)\}\,\mathrm{d}^2 z, \tag{A.10}$$

then

$$F(z,z^*) = \int \tilde{F}(\xi,\xi^*)\exp\{-\mathrm{i}(z\xi + z^*\xi^*)\}\,\mathrm{d}^2\xi. \tag{A.11}$$

- The Gamma function $\Gamma(z)$ is defined by

$$\Gamma(z) = \int_0^\infty \exp(-x)x^{z-1}\mathrm{d}x, \qquad \mathrm{Re}(z) > 0. \tag{A.12}$$

$$\Gamma(z+1) = z\Gamma(z). \tag{A.13}$$

If m a positive integer or zero then

$$\Gamma(m+1) = m!, \qquad \frac{1}{\Gamma(-m)} \to 0. \tag{A.14}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{A.15}$$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}. \tag{A.16}$$

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right). \tag{A.17}$$

- The beta function $B(\alpha, \beta)$ is defined by

$$B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\mathrm{d}x, \quad \mathrm{Re}(\alpha) > 0, \ \mathrm{Re}(\beta) > 0. \tag{A.18}$$

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{A.19}$$

- If x is a real variable and $\mathrm{Re}(\alpha) > 0$ then

$$\int_{-\infty}^{\infty} \mathrm{d}x \exp(-\alpha x^2 + \beta x) = \sqrt{\frac{\pi}{\alpha}} \exp(\beta^2/4\alpha), \tag{A.20}$$

$$\int_{-\infty}^{\infty} \mathrm{d}x \ x^{2m} \exp(-\alpha x^2) = \frac{1}{\sqrt{\alpha^{2m+1}}}\Gamma\left(m+\frac{1}{2}\right). \tag{A.21}$$

If $\hat{A}$ is an $N \times N$ symmetric matrix whose eigenvalues are positive then $\det(\hat{A}) > 0$ and

$$\int_{-\infty}^{\infty} \mathrm{d}x_1 \ldots \int_{-\infty}^{\infty} \mathrm{d}x_N \exp\left(-\boldsymbol{X}^T\hat{A}\boldsymbol{X} + \boldsymbol{C}^T\boldsymbol{X}\right)$$
$$= \sqrt{\frac{\pi^N}{\det(\hat{A})}} \exp\left[\frac{1}{4}\boldsymbol{C}^T\hat{A}^{-1}\boldsymbol{C}\right], \tag{A.22}$$

where $\boldsymbol{X}$ is a column constituted by the elements $x_1, \ldots, x_N$, $\boldsymbol{C}$ is a column of N constants and $\boldsymbol{C}^T$ denotes the transpose of $\boldsymbol{C}$.

- Integration over complex α plane may be carried by substituting $\alpha = x + \mathrm{i}y$, $(-\infty \leq x, y \leq \infty)$ so that

$$\int \mathrm{d}^2\alpha \to \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y. \tag{A.23}$$

It may alternatively be performed in the polar representation $\alpha = \sqrt{r}\exp(\mathrm{i}\theta)$ $(0 \leq r \leq \infty,\ 0 \leq \theta \leq 2\pi)$ so that

$$\int \mathrm{d}^2\alpha \to \frac{1}{2}\int_0^{\infty} dr \int_0^{2\pi} \mathrm{d}\theta. \tag{A.24}$$

Using this representation, it may be verified that, if $\mathrm{Re}(a) > 0$ then

$$\frac{1}{\pi}\int \mathrm{d}^2\alpha\ \alpha^m \alpha^{*n} \exp(-a|\alpha|^2) = \frac{m!}{a^{m+1}}\delta_{mn}. \tag{A.25}$$

$$\frac{1}{\pi}\int \mathrm{d}^2\alpha \exp(-|\alpha|^2 + \beta^*\alpha) f(\alpha^*) = f(\beta^*). \tag{A.26}$$

$$\frac{1}{\pi}\int \mathrm{d}^2\alpha \exp(-|\alpha|^2 + \beta^*\alpha)\alpha^m f(\alpha^*) = \frac{\partial^m}{\partial\beta^{*m}} f(\beta^*). \tag{A.27}$$

$$\frac{1}{\pi}\int \mathrm{d}^2\alpha \exp(-a|\alpha|^2 + b_1\alpha + b_2\alpha^*) = \frac{1}{a}\exp\left(\frac{b_1 b_2}{a}\right). \tag{A.28}$$

If $a^2 - 4|c|^2 > 0$ then

$$\begin{aligned}&\frac{1}{\pi}\int \mathrm{d}^2\alpha \exp\left(-a|\alpha|^2 + b\alpha + b^*\alpha^* + c\alpha^2 + c^*\alpha^{*2}\right)\\ &= \sqrt{\frac{1}{a^2 - 4|c|^2}}\exp\left[\frac{b^2 c^* + b^{*2}c + a|b|^2}{a^2 - 4|c|^2}\right].\end{aligned} \tag{A.29}$$

- The Hermite polynomial $H_n(x)$ is defined by

$$H_n(x) = (-)^n \exp(x^2)\frac{\mathrm{d}^n}{\mathrm{d}x^n}\exp(-x^2), \tag{A.30}$$

$$H_n(x) = \sum_{m=0}^{(n/2)} (-)^m \frac{n!}{m!(n-2m)!}(2x)^{n-2m}. \tag{A.31}$$

Its integral representation and the generating functions are

$$H_n(x) = \frac{2^n}{\sqrt{\pi}}\int_{-\infty}^{\infty} (x + \mathrm{i}t)^n \exp(-t^2)\mathrm{d}t, \tag{A.32}$$

$$\exp\left(-x^2 + 2xy\right) = \sum_{m=0}^{\infty} \frac{x^m}{m!} H_m(y). \tag{A.33}$$

- The spherical harmonics $Y_{LM}(\theta,\phi)$ are such that

$$Y_{LM}(\theta,\phi) = (-)^M Y^*_{L-M}(\theta,\phi), \tag{A.34}$$

$$Y_{00}(\theta,\phi) = \frac{1}{\sqrt{4\pi}}, \tag{A.35}$$

$$\int_0^\pi \sin(\theta)\mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi Y_{LM}(\theta,\phi) Y^*_{KQ}(\theta,\phi) = \delta_{KL}\delta_{MQ}, \tag{A.36}$$

$$\sum_{L=0}^{\infty}\sum_{M=-L}^{L} Y_{LM}(\theta',\phi') Y^*_{LM}(\theta,\phi)$$
$$= \delta(\phi-\phi')\delta(\cos(\theta)-\cos(\theta')). \tag{A.37}$$

- If $F(a,b;c;z)$ is the Gauss hypergeometric function then

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{A.38}$$

B. Hypergeometric Equation

In this Appendix we address the question of reducibility of a second-order differential equation

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + P(x)\frac{\mathrm{d}}{\mathrm{d}x} + Q(x)\right] f(x) = 0 \tag{B.1}$$

to an equation for the hypergeometric or the confluent hypergeometric function. For details see, for example, [149, 150].

The nature of solution of (B.1) is governed by its singularities. We recall that if $P(x)$ and $Q(x)$ are analytic at a point x_0 then x_0 is called an *ordinary point* of (B.1). The x_0 is a *singular point* of (B.1) if it is an isolated singularity of either $P(x)$ or $Q(x)$ or both. The singular point x_0 is called a *regular singularity* if $(x-x_0)P(x)$ and $(x-x_0)^2Q(x)$ are analytic at $x=x_0$. In other words, $x=x_0$ is a regular singularity if $P(x)$ has a pole of order no more than one and $Q(x)$ has a pole of order no more than two at x_0.

The nature of singularity at infinity is examined by transforming to $y = 1/x$. The eq.(B.1) in terms of $y = 1/x$ reads

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \left(\frac{2}{y} - \frac{P(1/y)}{y^2}\right)\frac{\mathrm{d}}{\mathrm{d}y} + \frac{Q(1/y)}{y^4}\right] f(y) = 0. \tag{B.2}$$

The singularity of this equation at $y=0$ corresponds to that of (B.1) at $x=\infty$.

Let x_0 be a regular singular point with

$$\lim_{x\to x_0}(x-x_0)P(x) = p_0, \quad \lim_{x\to x_0}(x-x_0)^2 Q(x) = q_0. \tag{B.3}$$

The equation

$$\gamma^2 - (1 - p_0)\gamma + q_0 = 0 \tag{B.4}$$

is called the *indicial equation* and its roots $\gamma_\pm$ the *exponents* or *indices* corresponding to x_0.

The equation (B.1) is called a *Riemann equation* if it has three regular singularities including the singularity at infinity. The sum of the exponents corresponding to those singularities is unity. By an appropriate transformation of $x \to z$, its three singular points can be mapped on to $z = 0, 1, \infty$. This, followed by a transformation of $f(x) \to w(z)$ transforms (B.1) to the *hypergeometric equation*

$$\left[z(1-z)\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \{c - (a+b+1)z\}\frac{\mathrm{d}}{\mathrm{d}z} - ab\right]w(z) = 0. \tag{B.5}$$

If c is not an integer then, within the unit circle $|z| < 1$, linearly independent solutions of (B.5) are $F(a, b; c; z)$ and $z^{1-c}F(a-c+1, b-c+1; 2-c; z)$ where $F(a, b; c; z)$ is the *Gauss hypergeometric function*

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{m=0} \frac{\Gamma(m+a)\Gamma(m+b)}{\Gamma(m+c)\Gamma(m+1)} z^m. \tag{B.6}$$

Note that if $a = -m$ or $b = -m$ where m is a positive integer then $F(a, b; c; z)$ is a polynomial of degree m.

The equation (B.5) has $z = 0, 1, \infty$ as its regular singular points the sum of whose indices is 1. In addition to this, one of the indices of each of the singularities at $z = 0, 1$ is zero. The transformations that reduce a Riemann equation to a hypergeometric equation are known [150]. However, we outline below the procedure for transforming (B.1) to (B.5) only for the particular forms of $P(x)$, $Q(x)$ encountered in Chap. 10.

A form of (B.1) that concerns us is

$$\left[x(1-x)^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} + p(1-x)(x+q)\frac{\mathrm{d}}{\mathrm{d}x} + (rx - \lambda)\right]f(x) = 0, \tag{B.7}$$

p, q, r, λ being constants. Verify that

$$f(x) = (1-x)^\alpha w(x) \tag{B.8}$$

transforms (B.7) to an equation for $w(x)$ which is of the form (B.5) if

$$\alpha(\alpha - 1) - \alpha p + r = \alpha p q + \lambda, \tag{B.9}$$

with a, b determined by solving

$$a + b + 1 = 2\alpha - p, \qquad ab = \alpha(\alpha - 1) - p\alpha + r, \tag{B.10}$$

and

$$c = pq. \tag{B.11}$$

A solution of (B.7) thus reads

$$f(x) = (1-x)^{\alpha} F(a,b;c;x), \tag{B.12}$$

with α and a, b, c determined by (B.9)-(B.11).

Confluent Hypergeometric Function. Consider the hypergeometric equation (B.5). Rewrite it by changing the independent variable z to $x = bz$. The singular points of the transformed equation are then at $x = 0, b, \infty$. Let $b \to \infty$ so that the singularity at $x = b$ merges with that at $x = \infty$. The transformed equation

$$\left[x\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (c-x)\frac{\mathrm{d}}{\mathrm{d}x} - a\right] f(x) = 0, \tag{B.13}$$

is called the *confluent hypergeometric equation.* Note that this equation is thus obtained by the confluence of two singularities at infinity. Also, $x = 0$ is a regular singularity of (B.12) whereas its singularity at $x = \infty$ is irregular. If c is not zero or a negative integer then a solution of this equation is the *confluent hypergeometric function*

$$\Phi(a;c;z) = \frac{\Gamma(c)}{\Gamma(a)} \sum_{m=0} \frac{\Gamma(m+a)}{\Gamma(m+c)\Gamma(m+1)} z^m. \tag{B.14}$$

Note that $\Phi(-m;c;x)$ is a polynomial of degree m.

C. Solution of Two- and Three-Dimensional Linear Equations

In this Appendix we derive exact solution of a linear equation

$$\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = \hat{M}|\psi(t)\rangle + |S(t)\rangle \tag{C.1}$$

when $\hat{M}$ is a t-independent 2×2 or a 3×3 matrix. Its formal solution is given by

$$|\psi(t)\rangle = \exp(\hat{M}t)\left[|\psi(0)\rangle + \int_0^t \mathrm{d}\tau \exp(-\hat{M}\tau)|S(\tau)\rangle\right]. \tag{C.2}$$

If the real part of all the eigenvalues of $\hat{M}$ is negative then, in the limit $t \to \infty$, $\exp(\hat{M}t) \to 0$. If, in addition, $|S\rangle$ is independent of time then (C.2) yields

$$|\psi(\infty)\rangle = -\hat{M}^{-1}|S\rangle. \tag{C.3}$$

Consider first the case of two-dimensional $\hat{M}$ given by

$$\hat{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{C.4}$$

Let λ_1 and λ_2 be the eigenvalues of $\hat{M}$. Assume that $\lambda_1 \neq \lambda_2$ and invoke (10.12) to obtain

$$\exp(\hat{M}t) = \frac{1}{\lambda_1 - \lambda_2}\left[(\hat{M} - \lambda_2)\exp(\lambda_1 t) - (\hat{M} - \lambda_1)\exp(\lambda_2 t)\right]$$
$$= \frac{1}{\lambda_1 - \lambda_2}\begin{pmatrix} \beta_{11}(t) & \beta_{12}(t) \\ \beta_{21}(t) & \beta_{22}(t) \end{pmatrix}, \tag{C.5}$$

$$\beta_{11}(t) = (a - \lambda_2)\exp(\lambda_1 t) - (a - \lambda_1)\exp(\lambda_2 t),$$
$$\beta_{22}(t) = (d - \lambda_2)\exp(\lambda_1 t) - (d - \lambda_1)\exp(\lambda_2 t),$$
$$\beta_{12}(t) = b(\exp(\lambda_1 t) - \exp(\lambda_2 t)),$$
$$\beta_{21}(t) = c(\exp(\lambda_1 t) - \exp(\lambda_2 t)). \tag{C.6}$$

Combination of (C.2), (C.5) and (C.6) yields the solution of (C.1). Its limit $\lambda_1 \to \lambda_2$ gives the solution for $\lambda_1 = \lambda_2$.

Next, let $\hat{M}$ be a 3×3, t-independent matrix. Let $\lambda_1, \lambda_2, \lambda_3$ be its eigenvalues assumed to be distinct. Invoke (10.12) to obtain

$$\exp\left(\hat{M}\ t\right) = \sum_{m=1}^{3} \exp(\lambda_m t) \prod_{n \neq m} \frac{\hat{M} - \lambda_n}{\lambda_m - \lambda_n}$$
$$= \sum_{m=1}^{3} \alpha_m(t)\hat{M}^{m-1}, \tag{C.7}$$

the $\alpha_m(t)$ being unknown functions. The second line in the equation above is owing to the fact that the maximum power of $\hat{M}$ in the first line is two. The $\alpha_m(t)'s$, obtained by comparing equal powers of $\hat{M}$ in the two equations in (C.7), are found to be given by

$$\begin{pmatrix} \alpha_1(t) \\ \alpha_2(t) \\ \alpha_3(t) \end{pmatrix} = \frac{1}{D}\begin{pmatrix} \lambda_2\lambda_3(\lambda_3 - \lambda_2) & \lambda_1\lambda_3(\lambda_1 - \lambda_3) & \lambda_1\lambda_2(\lambda_2 - \lambda_1) \\ \lambda_3^2 - \lambda_2^2 & \lambda_1^2 - \lambda_3^2 & \lambda_2^2 - \lambda_1^2 \\ \lambda_3 - \lambda_2 & \lambda_1 - \lambda_3 & \lambda_2 - \lambda_1 \end{pmatrix}$$
$$\times \begin{pmatrix} \exp(\lambda_1\ t) \\ \exp(\lambda_2\ t) \\ \exp(\lambda_3\ t) \end{pmatrix}, \tag{C.8}$$

$$D = (\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1). \tag{C.9}$$

The $\exp(\hat{M}t)$ is determined by inserting (C.8) in (C.7). The resulting expression, on combination with (C.2), determines the solution of (C.1).

D. Roots of a Polynomial

In this Appendix we list some general properties of the roots of a polynomial and the exact expression for the roots of a cubic. For details, see [151].

1. Consider the polynomial equation

$$f_n(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda^1 + a_0 = 0 \tag{D.1}$$

where $a_0, \ldots, a_n$ are real constants. This equation admits n roots $\lambda_1, \ldots, \lambda_n$. The roots may be real or complex. However, if λ_i is a root then, since $\{a_k\}$ are real, it follows that $f_n^*(\lambda_i) = f_n(\lambda_i^*) = 0$. Hence λ_i^* is also a root.

2. If $f(a)$ and $f(b)$ are of opposite sign then $f_n(\lambda) = 0$ has a root in the interval (a, b).

3. If one of the roots of a polynomial $f_n(\lambda) = 0$ is known to be, say, λ_1 then on dividing that polynomial by $\lambda - \lambda_1$ we obtain the quotient which is a plynomial $\phi_{n-1}(\lambda)$ of degree $n-1$. The other roots of $f_n(\lambda) = 0$ are the roots of $\phi_{n-1}(\lambda) = 0$.

4. In the problems concerned with dynamical stability, it is crucial to know whether or not the roots are positive and the conditions under which the real part of the roots is negative. This task is helped by the following:

   - *Descarte's rule of signs*: If r_+ is the number of positive real roots of (D.1) and V is the number of changes of sign in the sequence of the coefficients $a_n, a_{n-1}, a_{n-2}, \ldots, a_0$ then
   $$r_+ = V - 2h, \tag{D.2}$$
   the h being a nonnegative integer. In other words, the maximum value of r_+ is V and, if less, always by an even number.
   - *Hurwitz criterion*: The real part of every root of (D.1) is negative if, and only if, $a_0 > 0$ and n determinants
   $$D_1 = a_1, \quad D_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix}, \quad D_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix}, \cdots,$$
   $$D_n = \begin{vmatrix} a_1 & a_0 & \cdots & 0 \\ a_3 & a_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ a_{2n-1} & a_{2n-2} & \cdots & a_n \end{vmatrix} \tag{D.3}$$
   are all positive.

5. Consider the cubic equation

$$f_3(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0. \tag{D.4}$$

If λ_1 is its root then, following the item number 3 above, we find that its other two roots are the solutions of the quadratic

$$\lambda^2 + (\lambda_1 + a_2)\lambda + a_1 + \lambda_1(\lambda_1 + a_2) = 0. \tag{D.5}$$

In order to list the roots of (D.4), we define

$$p = a_1 - \frac{1}{3}a_2^2, \quad q = a_0 - \frac{1}{3}a_1a_2 + \frac{2}{27}a_2^3, \tag{D.6}$$

$$A = -\frac{q}{2} + \sqrt{\Delta}, \qquad B = -\frac{q}{2} - \sqrt{\Delta}, \tag{D.7}$$

$$\Delta = \frac{q^2}{4} + \frac{p^3}{27}. \tag{D.8}$$

The roots of (D.4) are

$$\begin{aligned}
\lambda_1 &= A^{1/3} + B^{1/3} - \frac{1}{3}a_2, \\
\lambda_2 &= -\frac{A^{1/3} + B^{1/3}}{2} + \mathrm{i}\sqrt{3}\frac{A^{1/3} - B^{1/3}}{2} - \frac{1}{3}a_2, \\
\lambda_2 &= -\frac{A^{1/3} + B^{1/3}}{2} - \mathrm{i}\sqrt{3}\frac{A^{1/3} - B^{1/3}}{2} - \frac{1}{3}a_2.
\end{aligned} \tag{D.9}$$

Now, if $\Delta < 0$ then A, B are complex conjugate of each other. Hence all the three roots are real if $\Delta < 0$. If $\Delta > 0$ then A, B are real. Hence, if $\Delta > 0$ then one root (λ_1) is real and the other two are complex conjugate of each other.

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